MA40042 Measure Theory and Integration

Conventions regarding infinity

At times in this course, we will want to treat ∞ (which we sometimes write as $+\infty$) and even $-\infty$ similarly to real numbers. For example the 'measure' (size) of a set can be ∞, and the definite integral of a function can be $\pm\infty$. Also, it often make sense to say that a diverging sequence or function converges to $\pm\infty$.

Most of the arithmetic with $\pm\infty$ is common sense, but occasionally you need to be a little careful. For completeness, the rules are these:

- For $a \in \mathbb{R}$, we set $a + \infty = \infty + a = \infty$ and $a + (-\infty) = -\infty + a = -\infty$.
- $\infty + \infty = \infty$ and $-\infty + (-\infty) = -\infty$.
- $\infty + (-\infty)$ and $-\infty + \infty$ are not defined.
- For $a \in \mathbb{R}$ with $a > 0$, we set $a \cdot \infty = \infty \cdot a = \infty$ and $a \cdot (-\infty) = (-\infty) \cdot a = -\infty$.
- For $a \in \mathbb{R}$ with $a < 0$, we set $a \cdot \infty = \infty \cdot a = -\infty$ and $a \cdot (-\infty) = (-\infty) \cdot a = \infty$.
- $\infty \cdot \infty = \infty$, $(-\infty) \cdot \infty = \infty \cdot (-\infty) = -\infty$ and $(-\infty) \cdot (-\infty) = \infty$.
- $0 \cdot \infty = \infty \cdot 0 = 0$ and $0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$.

This last one is unusual (and does not necessarily hold outside of this course). The idea is the two-dimension area of the box $[a, b) \times [c, d)$ should be the product of the lengths $(b-a)(d-c)$. By this logic, the 2D area of the infinite line $\{0\} \times (-\infty, \infty)$ should be $0 \times \infty$, but we want this area to be 0.

Remember to be careful with cancellations when ∞ is involved. If $a + b = a + c$, then $b = c$, *provided that* $a \neq \pm\infty$. If $ab = ac$, then $b = c$, *provided that* $a \neq -\infty, 0, \infty$.

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