Carathéodory's splitting condition

So far, starting from X, $\mathcal{R}$ and ρ, we have constructed an outer measure μ^* on all of $\mathcal{P}(X)$. But μ^* is not, in general, a measure. However, we shall see that if we restrict to a smaller σ-algebra, then the restriction of μ^* to this σ-algebra *is* a measure.

Specifically, we restrict to those sets A such that

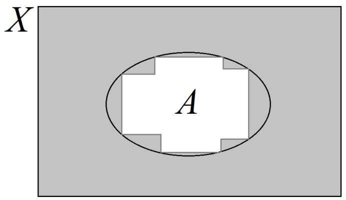
$$\boxed{\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c) \qquad \text{for all } S \subset X}$$

Let's try and give some motivation for this (rather opaque) condition.

The natural idea is to form an 'inner measure' μ_* by approximating a set A from the inside with sets from X, and defining A to be measurable if the inner measure and outer measure agree, $\mu_*(A) = \mu^*(A)$. Indeed, this is what Lebesgue did. But it's fiddlier than you'd hope for the Lebesgue measure, and very tricky to generalise. Thus we define measurable sets be those satisfying a certain 'splitting condition', which we aim to justify here.

One could instead think taking the inner measure of A as being like taking the outer measure of $A^c = X \setminus A$. That is, one could define

$$\mu_*(A) = \mu^*(X) - \mu^*(A^c).$$

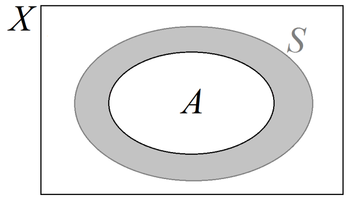


Then the inner and outer measures being equal would be the condition that

$$\mu^*(A) = \mu^*(X) - \mu^*(A^c).$$

While this works for finite measure spaces, we get into trouble when $\mu^*(X)$ is infinite. We might then think of defining the 'inner measure with respect to S', for $A \subset S \subset X$, as

$$\mu_S(A) = \mu^*(S) - \mu^*(S \setminus A).$$

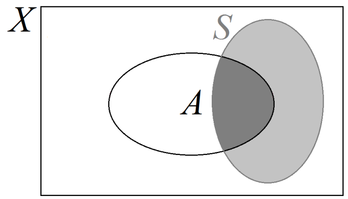


The 'inner = outer' condition becomes

$$\mu^*(A) = \mu^*(S) - \mu^*(S \setminus A) \qquad \text{for all } S \subset X, S \supset A\ .$$

In fact, it's better to let S not be a superset of A, but to 'cut out' the part where it overlaps, so

$$\mu_S(S \cap A) = \mu^*(S) - \mu^*(S \cap A^c).$$



Thus we get the condition

$$\mu^*(S \cap A) = \mu^*(S) - \mu^*(S \cap A^c) \qquad \text{for all } S \subset X.$$

Finally, we rewrite this to avoid '∞ minus ∞' difficulties, to get the *splitting condition*

$$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c) \qquad \text{for all } S \subset X.$$

Definition. Let X be a nonempty set, and μ^* be an outer measure on X. We say that a set $A \subset X$ is *Carathéodory measurable with respect to* μ^* (or just *measurable* for short) if it satisfies *Carathéodory's splitting condition*:

$$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c) \qquad \text{for all } S \subset X.$$

We write $\mathcal{M}$ for the collection of measurable sets, and μ for the restriction of μ^* to $\mathcal{M}$; that is, the function $\mu\colon \mathcal{M} \to [0,\infty]$ with $\mu(A) = \mu^*(A)$ for $A \in \mathcal{M}$.

Theorem. *Let X be a nonempty set, μ^* an outer measure on X, and $\mathcal{M}$ the collection of Carathéodory measurable sets with respect to μ^*. Write μ for the restriction of μ^* to μ. Then $(X, \mathcal{M}, \mu)$ is a measure space.*