MA40042 Measure Theory and Integration

Constructing measures: An outline

0. Setup

Let X be a nonempty set. We start with

- A collection $\mathcal{R}$ of subsets of X, with $\varnothing \in \mathcal{R}$.
- A function $\rho\colon \mathcal{R} \to [0,\infty]$, with $\rho(\varnothing) = 0$.

The idea is that $\mathcal{R}$ is a collection of 'simple' sets, and we want the measure of $R \in \mathcal{R}$ to be $\rho(R)$.

An important case is the *Lebesgue measure*. On $X = \mathbb{R}$, we take $\mathcal{R}$ to be the collection of intervals $\mathcal{I}$, and define the length ρ by

$$\rho(\varnothing) = 0 \qquad \rho\big([a,b)\big) = b - a \qquad \rho\big([-\infty,b)\big) = \rho\big([a,\infty)\big) = \rho(\mathbb{R}) = \infty.$$

1. Outer measure

Given a countable subcollection $\mathcal{C} \subset \mathcal{R}$, we write

$$\rho(\mathcal{C}) = \sum_{R\in\mathcal{C}} \rho(R).$$

We call $\mathcal{C}$ a *covering* of a set $A \subset X$ if $A \subset \bigcup_{R\in\mathcal{C}} R$.

We define a function $\mu^*\colon \mathcal{P}(X) \to [0,\infty]$ by

$$\mu^*(A) := \inf\{\rho(\mathcal{C}) : \mathcal{C} \text{ is a covering of } A\}.$$

The function μ^* is an *outer measure*, in that

1. $\mu^*(\varnothing) = 0$
2. μ^* is monotone, in that for $A \subset B \subset X$ we have $\mu^*(A) \leq \mu^*(B)$.
3. μ^* is countably subadditive, in that for a countably infinite sequence A_1, $A_2, \dots$ of subsets of X, we have

$$\mu^*\left(\bigcup_n A_n\right) \leq \sum_n \mu^*(A_n).$$

2. Measurable sets

We say that a set $A \subset X$ is *measurable* if it satisfies the splitting condition

$$\mu^*(S) = \mu^*(S\cap A) + \mu^*(S \cap A^{\mathsf{c}}) \qquad \text{for all } S \subset X.$$

We write $\mathcal{M}$ for the collection of measurable sets, and μ for the restriction of μ^* to $\mathcal{M}$.

The triple $(X, \mathcal{M}, \mu)$ is a measure space.

3. Carathéodory's extension theorem

A function $\pi\colon \mathcal{R} \to [0,\infty]$ is a *premeasure* on $(X,\mathcal{R})$ if

1. $\pi(\varnothing) = 0$;
2. if $A_1, A_2, \dots$ is a countable sequence of disjoint sets in $\mathcal{R}$ and if their union $\bigcup_{n=1}^\infty A_n$ is also in $\mathcal{R}$, then

$$\pi\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \pi(A_n).$$

A collection $\mathcal{S}$ of subsets of X is a *semialgebra* if

1. $\varnothing \in \mathcal{S}$;
2. $\mathcal{S}$ is closed under finite intersections, in that for $A, B \in \mathcal{S}$ we have $A \cap B \in \mathcal{S}$;
3. 'complements are finite disjoint unions,' in that for $A \in \mathcal{S}$, there exists disjoint $B_1, B_2, \dots, B_N$ in $\mathcal{S}$ such that $A^{\mathsf{c}} = \bigcup_{n=1}^N B_n$.

Carathéodory's extension theorem. Let X be a nonempty set, $\mathcal{S}$ be a semialgebra on X, and π a premeasure on $(X,\mathcal{S})$. Then there exists a measure μ which extends π.

Further, if μ is a σ-finite measure, then it is the unique such extension.

The *Lebesgue measure* on $\mathbb{R}$ is the unique measure λ on $(\mathbb{R},\mathcal{B})$ such that $\lambda\big([a,b)\big) = b - a$ for all $a < b$.

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