MA40042 Measure Theory and Integration

Lecture 1
σ-algebras

- σ-algebras: definition, examples, and basic properties
- Generated σ-algebras
- Algebras

1.1 Definition

Recall from last time the properties we wanted a collection of measurable sets in X to have:

1. The empty set $\varnothing$ is measurable.
2. Finite or countably infinite unions of measurable sets are measurable.
3. The set X itself is measurable.
4. Complements of measurable sets are measurable.

It's fairly easy to show (and we will later) that the following three points satisfy for a definition, and the other properties follow from these.

Definition 1.1. Let X be a nonempty set. A collection Σ of subsets of X is called a *σ-algebra* (or *σ-field*) on X, if it has the following properties:

1. $\varnothing \in \Sigma$;
2. if $A \in \Sigma$ then $A^{\mathsf{c}} := X \setminus A \in \Sigma$;
3. if $A_1, A_2, \dots$ is a countably infinite sequence of sets in Σ, then the union $\bigcup_{n=1}^{\infty} A_n \in \Sigma$ also.

The pair (X, Σ) is called a *measurable space.*

(Recall that in this course we are using the convention that $A \subset X$ means 'A is a subset of X or equal to X.' In particular, $X \subset X$ and $\varnothing \subset X$ are always true.)

In probability contexts, the set X is called the *sample space*, and is usually denoted Ω. The σ-algebra is usually denoted $\mathcal{F}$. The subsets A are called *events.* In point 2, A^{c} is the event that A does not occur; in point 3, the union is the event the at least one of the A_ns occurs.

1.2 Examples

The following are examples of σ-algebras. Checking that these really are σ-algebras is a homework exercise.

- For any nonempty set X the *powerset* of X is the set of all subsets of X; that is,
$$\mathcal{P}(X) = \{A : A \subset X\}.$$
The powerset is the most common σ-algebra on countable sets.

- For any nonempty set X, the *trivial σ-algebra* is $\Sigma = \{\emptyset, X\}$.

- Let X be a set and A a subset with $A \neq \varnothing, X$. Then $\Sigma = \{\emptyset, A, A^{\mathsf{c}}, X\}$ is a σ-algebra.

- Recall that a subset $A \subset X$ is called *countable* if it is finite or countably infinite, and is called *co-countable* f its complement A^{c} is countable. For nonempty X, the collection
$$\Sigma = \{A \subset X : A \text{ is countable or co-countable}\}$$
is a σ-algebra. (If X itself is countable, then $\Sigma = \mathcal{P}(X)$.)

1.3 Properties

Now let's go through some basic properties of σ-algebras.

Theorem 1.2. *Let (X, Σ) be a measurable space. Then we have the following:*

1. *$X \in \Sigma$;*
2. *if $A_1, A_2, \dots, A_N$ is a finite sequence of sets in Σ, then $\bigcup_{n=1}^{N} A_n \in \Sigma$ also;*
3. *if $A_1, A_2, \dots$ is a countably infinite sequence of sets in Σ, then the intersection $\bigcap_{n=1}^{\infty} A_n \in \Sigma$ also;*
4. *if $A_1, A_2, \dots, A_N$ is a finite sequence of sets in Σ, then $\bigcap_{n=1}^{N} A_n \in \Sigma$ also;*
5. *if $A, B \in \Sigma$ with $B \subset A$, then $B \setminus A \in \Sigma$;*
6. *if $A, B \in \Sigma$, then the symmetric difference $A \triangle B \in \Sigma$. (Recall that the* symmetric difference *$A \triangle B$ is the set of points in A or in B but not in both.)*

The idea is that that a σ-algebra is closed under countable sequences of set operations.

Proof. Most of these are left for homework, but we'll do 2 as an example.

First, recall *De Morgan's law*, that

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^{\mathsf{c}} \right)^{\mathsf{c}} .$$

(The right-hand side says it's not the case that you're missing from any of the sets A_n, which is equivalent to the left-hand side, which says you're in all of the sets A_n.)

Suppose the A_ns are all in Σ. By point 2 of the definition of a σ-algebra, their complements A_n^{c} are in Σ also. Then by point 3, their union $\bigcup_{n=1}^{\infty} A_n^{\mathsf{c}}$ is in Σ, and by point 2 the complement of that union is in Σ as well. By De Morgan's law, this complement of a union of complements is precisely the desired intersection. □

1.4 Generated σ-algebras

The following is an important definition.

Definition 1.3. Let X be a nonempty set, and let $\mathcal{C}$ be a collection of subsets of X. Then *the σ-algebra generated by $\mathcal{C}$*, denoted $\sigma(\mathcal{C})$, is the intersection of all σ-algebras on X containing $\mathcal{C}$; that is,

$$\sigma(\mathcal{C}) := \bigcap_{\substack{\Sigma \supset \mathcal{C} \\ \Sigma \text{ a } \sigma\text{-algebra on } X}} \Sigma. \qquad (*)$$

The reason it's important (and the reason for the name) comes from the following theorem.

Theorem 1.4. *Let X be a nonempty set, and let $\mathcal{C}$ be a collection of subsets of X. Then $\sigma(\mathcal{C})$ is a σ-algebra, and is the smallest σ-algebra containing $\mathcal{C}$.*

(By 'smallest', we mean that every other σ-algebra containing $\mathcal{C}$ is a proper superset of $\sigma(\mathcal{C})$.)

So if you want all the sets in some collection $\mathcal{C}$ to measurable, you have to at least use the σ-algebra generated by $\mathcal{C}$.

For example, if $\mathcal{C} = \{C\}$ contains a single set $C \neq \varnothing, X$, then $\sigma(\mathcal{C}) = \{\emptyset, C, C^{\mathsf{c}}, X\}$. If $\mathcal{C}$ is already a σ-algebra, then $\sigma(\mathcal{C}) = \mathcal{C}$.

Proof. Let's first check that $\sigma(\mathcal{C})$ is a σ-algebra. (Actually, we'll show that any intersection of σ-algebras is also a σ-algebra.) We need to show it satisfies the three criteria from the definition.

First, the empty set $\varnothing$ is in every σ-algebra containing $\mathcal{C}$, so is also in their intersection.

Second, if some set A is in every σ-algebra in the intersection, then its complement A^{c} is also in each of those σ-algebras, by point 2 of the definition. Hence A^{c} is also in the intersection.

Third, if each of the countable sequence of sets (A_n) is in every σ-algebra in the intersection, then their union $\bigcup_{n=1}^{\infty} A_n$ is also in each of those σ-algebras, by point 3 of the definition. Hence that union is also in the intersection. Hence the intersection is a σ-algebra, as desired.

Now we show that $\sigma(\mathcal{C})$ is the smallest σ-algebra containing $\mathcal{C}$.

That $\sigma(\mathcal{C})$ contains $\mathcal{C}$ is immediate, as it is intersection of collections that all contain $\mathcal{C}$.

To see it is the smallest such σ-algebra, let Σ be the smallest σ-algebra containing $\mathcal{C}$. But $\sigma(\mathcal{C})$ is by definition an intersection of collections including Σ, so is at least as small as Σ. □

1.5 Algebras

It's sometimes useful to have a looser definition than a σ-algebra. Since a σ-algebra is closed under countable set operations, it makes sense to define an *algebra* to be closed under just finite (but not necessarily countably infinite) set operations.

Definition 1.5. Let X be a nonempty set. A collection $\mathcal{A}$ of subsets of X is called an *algebra* (or *field*) on X, if it has the following properties:

1. $\varnothing \in \mathcal{A}$;
2. if $A \in \mathcal{A}$ then $A^{\mathsf{c}} \in \mathcal{A}$;
3. if $A_1, A_2, \dots, A_N$ is a finite sequence of sets in $\mathcal{A}$, then the union $\bigcup_{n=1}^{N} A_n \in \mathcal{A}$ also.

It's not hard to prove basic facts about algebras along the lines of Theorem 1.2. For example, $X \in \mathcal{A}$, and an algebra is closed under finite intersections.

An easy fact is the following.

Theorem 1.6. *Every σ-algebra is an algebra.*

Proof. Points 1 and 2 of the definition are immediate, and point 3 is shown by point 2 of Theorem 1.2. □

However, the converse is not true. A homework problem invites you to give an example of a σ-algebra that is not an algebra.

Another problem invites you to define $a(\mathcal{C})$, the algebra generated by $\mathcal{C}$.

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