MA40042 Measure Theory and Integration

Lecture 5
Constructing measures II: Measurable sets

- Carathéodory's splitting condition
- The measure space $(X, \mathcal{M}, \mu)$

5.1 Carathéodory's splitting condition

So far, starting from X, $\mathcal{R}$ and ρ, we have constructed an outer measure μ^* on all of $\mathcal{P}(X)$, which is monotone, countably subadditive, and has $\mu^*(\varnothing) = 0$. But μ^* is not, in general, a measure. However, we shall see that if we restrict to a smaller σ-algebra, then the restriction of μ^* to this σ-algebra *is* a measure.

Definition 5.1. Let X be a nonempty set, and μ^* be an outer measure on X. We say that a set $A \subset X$ is *Carathéodory measurable with respect to* μ^* (or just *measurable* for short) if it satisfies *Carathéodory's splitting condition*:

$$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^{\mathsf{c}}) \qquad \text{for all } S \subset X.$$

We write $\mathcal{M}$ for the collection of measurable sets, and μ for the restriction of μ^* to $\mathcal{M}$; that is, the function $\mu\colon \mathcal{M} \to [0,\infty]$ with $\mu(A) = \mu^*(A)$ for $A \in \mathcal{M}$.

(This splitting condition is quite mysterious. We aim to give some motivation for it in the 'Carathéodory's splitting condition' handout.)

Note that by finite subadditivity of μ^* we always have

$$\mu^*(S) \leq \mu^*(A \cap S) + \mu^*(A^{\mathsf{c}} \cap S),$$

since $S = (A \cap S) \cup (A^{\mathsf{c}} \cap S)$. Thus, to show measurability we only have to prove the opposite inequality

$$\mu^*(S) \geq \mu^*(A \cap S) + \mu^*(A^{\mathsf{c}} \cap S).$$

5.2 $(X, \mathcal{M}, \mu)$ is a measure space

Now the crucial result.

Theorem 5.2. *Let X be a nonempty set, μ^* an outer measure on X, and $\mathcal{M}$ the collection of Carathéodory measurable sets with respect to μ^*. Write μ for the restriction of μ^* to μ. Then $(X, \mathcal{M}, \mu)$ is a measure space.*

So we have to first show that $\mathcal{M}$ is a σ-algebra on X, and then that μ is a measure on $(X, \mathcal{M})$.

Let's begin with $\mathcal{M}$. We can start by showing it's an algebra.

Lemma 5.3. *The collection of measurable sets $\mathcal{M}$ is an algebra.*

Proof. The splitting condition for $A = \varnothing$ is $\mu^*(S) = \mu^*(\varnothing) + \mu^*(S)$, and since $\mu^*(\varnothing) = 0$, we see that the empty set is measurable.

The splitting condition remains the same under swapping A for A^{c}, so $\mathcal{M}$ is closed under complements.

Now for closure under finite unions. Since the splitting condition involves intersections, it will be simpler to prove closure under finite intersections. By De Morgan's law, the result for unions follows. By induction, we can just deal with the $N = 2$ case.

Let $A, B \in \mathcal{M}$ and $S \subset X$. We need to show that

$$\mu^*(S) \geq \mu^*(S \cap A \cap B) + \mu^*\big(S \cap (A \cap B)^{\mathsf{c}}\big).$$

First, since A satisfies the splitting condition, we have

$$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^{\mathsf{c}}). \tag{$*$}$$

Next, we can rewrite the term $\mu^*(S \cap A)$ using the splitting condition for B. Since B is measurable, using $S \cap A$ in place of S in the splitting condition, we have

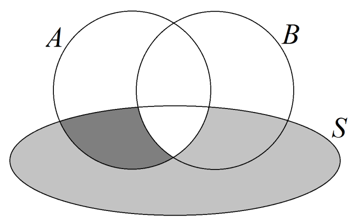
$$\mu^*(S \cap A) = \mu^*(S \cap A \cap B) + \mu^*(S \cap A \cap B^{\mathsf{c}}). \tag{$**$}$$

Substituting $(**)$ into $(*)$ gives

$$\mu^*(S) = \mu^*(S \cap A \cap B) + \mu^*(S \cap A \cap B^{\mathsf{c}}) + \mu^*(S \cap A^{\mathsf{c}}).$$

But, by drawing a picture we see that

$$(S \cap A \cap B^{\mathsf{c}}) \cup (S \cap A^{\mathsf{c}}) = S \cap (A \cap B)^{\mathsf{c}}.$$



Hence, since μ^* is finitely subadditive, we have that

$$\mu^*(S) \geq \mu^*(S \cap A \cap B) + \mu^*\big(S \cap (A \cap B)^{\mathsf{c}}\big),$$

and thus $A \cap B$ is measurable and we are done. □

The following is a useful lemma.

Lemma 5.4. *Let $A_1, A_2, \dots, A_N$ be a finite sequence of disjoint sets in $\mathcal{M}$. Then for all $S \subset X$, we have*

$$\mu^* \left(S \cap \bigcup_{n=1}^{N} A_n \right) = \sum_{n=1}^{N} \mu^*(S \cap A_n).$$

Proof. Again, it suffices to prove for $N = 2$.

Suppose $A, B \in \mathcal{M}$ are disjoint. The left-hand side of the statement in the lemma is $\mu^*\big(S \cap (A \cup B)\big)$. Using the splitting condition for A with $S \cap (A \cup B)$ taking the place of S, we can write this as

$$\begin{aligned} \mu^*\big(S \cap (A \cup B)\big) &= \mu^*\big(S \cap (A \cup B) \cap A\big) + \mu^*\big(S \cap (A \cup B) \cap A^{\mathsf{c}}\big) \\ &= \mu^*(S \cap A) + \mu^*(S \cap B), \end{aligned}$$

where we have used that A and B are disjoint. □

We can now prove that μ is finitely additive, which is part way to showing it is a measure.

Lemma 5.5. *The measure μ is finitely additive on $\mathcal{M}$. That is, if $A_1, A_2, \dots, A_N$ is a finite sequence of disjoint sets in $\mathcal{M}$, then*

$$\mu \left(\bigcup_{n=1}^{N} A_n \right) = \sum_{n=1}^{N} \mu(A_n).$$

Proof. Set $S = X$ in the Lemma 5.4, recalling that $\mu^* = \mu$ on $\mathcal{M}$. □

Now we're ready to complete the work.

Lemma 5.6. *The collection of measurable sets $\mathcal{M}$ is a σ-algebra on X.*

Proof. Since $\mathcal{M}$ is an algebra, we only have to show closure under countable unions.

Let $A_1, A_2, \dots$ be a countably infinite sequence of sets in $\mathcal{M}$. In order to apply the previous lemmas, we 'disjointify' by setting

$$B_1 = A_1, \qquad B_N = A_N \setminus \bigcup_{n=1}^{N-1} A_n \quad \text{for } N \geq 2.$$

Note that the B_n are measurable, since $\mathcal{M}$ is an algebra, and that $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n$. So we need to show that

$$\mu^*(S) \geq \mu^* \left(S \cap \bigcup_{n=1}^{\infty} B_n \right) + \mu^* \left(S \cap \left(\bigcup_{n=1}^{\infty} B_n \right)^{\mathsf{c}} \right).$$

Since $\mathcal{M}$ is an algebra, we have for any N,

$$\begin{aligned} \mu^*(S) &\geq \mu^* \left(S \cap \bigcup_{n=1}^{N} B_n \right) + \mu^* \left(S \cap \left(\bigcup_{n=1}^{N} B_n \right)^{\mathsf{c}} \right) \\ &\geq \sum_{n=1}^{N} \mu^*(S \cap B_n) + \mu^* \left(S \cap \left(\bigcup_{n=1}^{\infty} B_n \right)^{\mathsf{c}} \right), \end{aligned}$$

where we have used Lemma 5.4 on the first term and monotonicity on the second term. Since this holds for all N, we can send N to ∞. Thus

$$\begin{aligned} \mu^*(S) &\geq \sum_{n=1}^{\infty} \mu^*(S \cap B_n) + \mu^* \left(S \cap \left(\bigcup_{n=1}^{\infty} B_n \right)^{\mathsf{c}} \right) \\ &\geq \mu^* \left(\bigcup_{n=1}^{\infty} (S \cap B_n) \right) + \mu^* \left(S \cap \left(\bigcup_{n=1}^{\infty} B_n \right)^{\mathsf{c}} \right) \\ &= \mu^* \left(S \cap \bigcup_{n=1}^{\infty} B_n \right) + \mu^* \left(S \cap \left(\bigcup_{n=1}^{\infty} B_n \right)^{\mathsf{c}} \right), \end{aligned}$$

where we have used the countable subadditivity of μ^*. This proves the result. □

Lemma 5.7. *Let μ be the restriction of an outer measure μ^* on X to the measurable sets $\mathcal{M}$. Then μ is a measure on $(X, \mathcal{M})$.*

Proof. By definition $\mu(\varnothing) = \mu^*(\varnothing) = 0$. We only have to show countable additivity.

Let $A_1, A_2, \dots$ be a countably infinite sequence of disjoint measurable sets. Since outer measures are countably subadditive, we automatically have

$$\mu \left(\bigcup_{n=1}^{\infty} A_n \right) = \mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

It remains to prove the inequality in the other direction.

By Lemma 5.5 and monotonicity, we have for any N that

$$\sum_{n=1}^{N} \mu(A_n) = \mu \left(\bigcup_{n=1}^{N} A_n \right) \leq \mu \left(\bigcup_{n=1}^{\infty} A_n \right),$$

so we can send $N \to \infty$ to get the result. □

Together, Lemmas 5.6 and 5.7 prove that $(X, \mathcal{M}, \mu)$ is a measure space.