Lecture 6

Constructing measures III: Carathéodory's extension theorem

- Premeasures and semialgebras
- Carathéodory's extension theorem for algebras and for semialgebras.
- Lebesgue measure: existence and uniqueness

6.1 Carathéodory's extension theorem for algebras

The story so far:

0. We started with a set X and a collection $\mathcal{R}$ of sets, with a function ρ on $\mathcal{R}$, representing the sets whose measure we 'know'.
1. We constructed an outer measure μ^* on all of $\mathcal{P}(X)$.
2. We saw that if we restrict μ^* to μ on just the measurable sets $\mathcal{M}$ (satisfying the splitting condition), then μ is a measure on $\mathcal{M}$.

However, we haven't guaranteed that this constructed measure μ *extends* ρ, in the sense that all of $\mathcal{R}$ is measurable, and $\mu(R) = \rho(R)$ for $R \in \mathcal{R}$. In fact, in general, it's not true. However, it *is* true if $\mathcal{R}$ and ρ have certain properties.

First let's deal with ρ. Clearly ρ can't by itself contradict the measure axioms, as then an extension would have no hope.

Definition 6.1. Let X be a nonempty set, and let $\mathcal{R}$ be collection of subsets of X containing $\varnothing$. Then a function $\pi\colon \mathcal{R} \to [0,\infty]$ is a *premeasure* on $(X,\mathcal{R})$ if

1. $\pi(\varnothing) = 0$;
2. if $A_1, A_2, \dots$ is a countable sequence of disjoint sets in $\mathcal{R}$ and if their union $\bigcup_{n=1}^\infty A_n$ is also in $\mathcal{R}$, then
$$\pi\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \pi(A_n).$$

We will also need $\mathcal{R}$ to have some of the structure of a σ-algebra. An algebra is an example of this we already know.

Theorem 6.2 (Carathéodory's extension theorem for algebras)**.** *Let X be a nonempty set, $\mathcal{A}$ be an algebra on X, and π a premeasure on $(X,\mathcal{A})$. Then there exists a measure μ which extends π, in the sense that μ is a measure on $\big(X,\sigma(\mathcal{A})\big)$ with $\mu(A) = \pi(A)$ for $A \in \mathcal{A}$.*

Further, if μ is a σ-finite measure, then it is the unique such extension.

The idea is to take μ to be the restriction of the outer measure μ^* constructed via the covering method. So to prove the extension theorem we need to show that

- $\sigma(\mathcal{A}) \subset \mathcal{M}$, the measurable sets – since $\mathcal{M}$ is a σ-algebra, just showing $\mathcal{A} \subset \mathcal{M}$ suffices;
- $\mu(A) = \pi(A)$ for $A \in \mathcal{A}$;
- uniqueness (in the σ-finite case).

We shall do the proof later.

Since $\mathcal{M}$ is a complete measure space (see Problem Sheet 3), we could extend π even further to the completion $\big(X, \overline{\sigma(\mathcal{A})}, \bar\mu\big)$ of $\big(X,\sigma(\mathcal{A}),\mu\big)$ if we wished, but we won't bother in this course.

6.2 Carathéodory's extension theorem for semialgebras

While Theorem 6.2 is an important result, asking for $\mathcal{R}$ to be an algebra is quite a strenuous requirement. For example, the collection of intervals $\mathcal{I}$ we have in the setup for the Lebesgue measure is not an algebra, so this theorem is insufficient to prove existence of the Lebesgue measure.

Instead, we will look at a weaker definition.

Definition 6.3. Let X be a nonempty set, and $\mathcal{S}$ a collection of subsets of X. Then $\mathcal{S}$ is a *semialgebra* if

1. $\varnothing \in \mathcal{S}$;
2. $\mathcal{S}$ is closed under finite intersections, in that for $A, B \in \mathcal{S}$ we have $A \cap B \in \mathcal{S}$;
3. 'complements are finite disjoint unions,' in that for $A \in \mathcal{S}$, there exists disjoint $B_1, B_2, \dots, B_N$ in $\mathcal{S}$ such that $A^{\mathsf{c}} = \bigcup_{n=1}^N B_n$.

Theorem 6.4. *Every algebra is a semialgebra.*

Proof. The empty set is immediate, completements as finite disjoint unions from setting $B_1 = A^{\mathsf{c}}$, and intersections follows from De Morgan's law. □

Theorem 6.5 (Carathéodory's extension theorem for semialgebras)**.** *Let X be a nonempty set, $\mathcal{S}$ be a semialgebra on X, and π a premeasure on $(X, \mathcal{S})$. Then there exists a measure μ which extends π.*

Further, if μ is a σ-finite measure, then it is the unique such extension.

Again, we postpone the proof.

6.3 The Lebesgue measure

Back in Lecture 3, we defined the Lebesgue measure on $\mathbb{R}$ as follows.

Definition 6.6. The Lebesgue measure on $\mathbb{R}$ is the unique measure λ on $(\mathbb{R}, \mathcal{B})$ such that $\lambda\big([a, b)\big) = b - a$ for all $a < b$.

The Lebesgue measure on $\mathbb{R}^d$ is the unique measure λ on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$ such that

$$\lambda\left(\prod_{i=1}^{d}[a_i, b_i)\right) = \prod_{i=1}^{d}(b_i - a_i) \qquad \text{for all } a_i < b_i,\ i = 1, 2, \dots, d.$$

Theorem 6.7. *The Lebesgue measure exists and is unique.*

We shall give the full proof just for the $d = 1$ case, although the general case is much the same. Also the product measure (see Problem Sheet 3, and later in the course) gives an alternative construction in the $d \geq 2$ case.

Proof. First some housekeeping. All the intervals $[a, b)$ must be measurable, as must the empty set, while taking countable unions shows the infinite intervals $(\infty, b), [a, \infty), \mathbb{R}$ must be measurable too. This gives all the intervals in $\mathcal{I}$. Further, by countable additivity, the infinte intervals must have measure ∞, and the empty set must have measure 0. This gives the length function ρ as defined in the 'setup' of Lecture 4.

Since $\sigma(\mathcal{I}) = \mathcal{B}$, Carathéodory's extension theorem will show that the restriction the Lebesgue outer measure λ^* to $\mathcal{B}$ works. We just need to show that $\mathcal{I}$ is a σ-algebra, that the length ρ is a premeasure, and that λ is σ-finite. We prove these in the upcoming lemmas. □

Lemma 6.8. *The collection of intervals $\mathcal{I}$ is a semialgebra on $\mathbb{R}$.*

Proof. That $\varnothing \in \mathcal{I}$ is immediate. For finite intersections, note that

$$[a, b) \cap [c, d) = \big[\max\{a, c\}, \min\{b, d\}\big)$$

(with the latter interval interpreted as $\varnothing$ where necessary), with a similar result for the infinite and empty intervals. For complements, we have

$$[a, b)^{\mathsf{c}} = (-\infty, a) \cup [b, \infty),$$

and similar for the infinite and empty intervals. □

Basically the same proof works in d dimensions, although it takes a few lines to show the complement of an interval box in $\mathbb{R}^d$ can be written as a union of (at most) $2d$ interval boxes.

Lemma 6.9. *The 'length' function ρ is a premeasure on $(\mathbb{R}, \mathcal{I})$.*

Proof. We certainly have $\rho(\varnothing) = 0$.

We need to show countable additivity. Let $I_1, I_2, , \dots$ be disjoint intervals whose union is also an interval I. For finite N, we clearly have $\rho(I) \geq \sum_{n=1}^{N} \rho(I_n)$, for example by sorting the intervals from left to right. Sendin $N \to \infty$ gives $\rho(I) \geq \sum_{n=1}^{N} \rho(I_n)$.

We have to show the inequality the other way.

First, we assume the intervals are non-infinite, so the $I_n = [a_n, b_n)$ are disjoint, and their union is $I = [a, b)$. We use a compactness argument, which allows us to reduce from infinitely many to finitely many sets. Extend each interval to an open interval $I'_n = (a_n - \epsilon/2^n, b_n)$ (noting an upcoming use of the $\epsilon/2^n$ trick). Then these intervals cover the compact interval $[a, b - \epsilon]$ of length $b - a - \epsilon$. Since this is compact, the open cover $\{I'_1, I'_2, \dots\}$ has a finite subcover $\{I'_{n_1}, I'_{n_2}, \dots, I'_{n_k}\}$. Thus

$$b - a - \epsilon \leq \sum_{j=1}^{k} \rho(I'_{n_j}) \leq \sum_{j=1}^{k} \left(\rho(I_{n_j}) + \frac{\epsilon}{2^{n_j}}\right) \leq \sum_{j=1}^{k} \rho(I_{n_j}) + \epsilon \leq \sum_{n=1}^{\infty} \rho(I_n) + \epsilon.$$

Hence $\rho(I) \leq \sum_{n=1}^{\infty} \rho(I_n) + 2\epsilon$, and since ϵ was arbitrary, this proves the inequality.

Suppose instead that I is infinite. Then for fixed M, the intervals $I_n \cap [-M, M)$ are disjoint with union $I \cap [-M, M)$. By the previous paragraph, we see that

$$\sum_{n=1}^{\infty} \rho(I_n) \geq \sum_{n=1}^{\infty} \rho\big(I_n \cap [-M, M)\big) \geq \rho\big(I \cap [-M, M)\big).$$

The right-hand side tends to ∞ as $M \to \infty$, so the left-hand side must equal ∞ also. This gives the result. □

Lemma 6.10. *The Lebesgue measure λ on $\mathbb{R}$ is σ-finite.*

Proof. We have the countable union $\mathbb{R} = \displaystyle\bigcup_{n=1}^{\infty} [-n, n)$ with $\lambda([-n, n) = 2n \leq \infty$. □

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