Lecture 7

Proof of Carathéodory's extension theorem

- Proof of existence of extension for algebras and semialgebras

7.1 Proof of existence of extension for algebras

Recall Carathéodory's extension theorem for algebras. We'll just deal with existence of the extension here, and leave uniqueness for next time.

Theorem 7.1. *Let X be a nonempty set, $\mathcal{A}$ be an algebra on X, and π a premeasure on $(X, \mathcal{A})$. Then there exists a measure μ which extends π to $\big(X, \sigma(\mathcal{A})\big)$.*

Proof. Let μ^* be the outer measure on X constructed via the covering method, and let μ be its restriction to $\sigma(\mathcal{A})$. We know that μ^* is a measure when restricted to the collection $\mathcal{M}$ of Carathéodory measurable sets. Hence we need to show:

1. $\sigma(\mathcal{A}) \subset \mathcal{M}$. For this it suffices to show that $\mathcal{A} \subset \mathcal{M}$.

2. For $A \in \mathcal{A}$ we have $\mu^*(A) = \pi(A)$.

For point 1, we need to show that every $A \in \mathcal{A}$ satisfies the splitting condition

$$\mu^*(S) \geq \mu^*(S \cap A) + \mu^*(S \cap A^{\mathsf{c}}) \qquad \text{for all } S \subset X.$$

Recalling that we always have $\mu^*(S) \leq \mu^*(S \cap A) + \mu^*(S \cap A^{\mathsf{c}})$, we need only prove the opposite inequality.

For any $S \subset X$, by definition of the outer measure, we can find a covering $\{C_1, C_2, \dots\}$ of S with $\mu^*(S) + \epsilon \geq \sum_{n=1}^{\infty} \pi(C_n)$ for any $\epsilon > 0$. Now fix $A \in \mathcal{A}$. Since $\mathcal{A}$ is an algebra, $\{C_n \cap A : n \in \mathbb{N}\}$ and $\{C_n \cap A^{\mathsf{c}} : n \in \mathbb{N}\}$ are coverings in $\mathcal{A}$ of $S \cap A$ and $S \cap A^{\mathsf{c}}$ respecitvely. Since π is a premeasure, we thus have

$$\begin{aligned}
\mu^*(S) + \epsilon &\geq \sum_{n=1}^{\infty} \pi(C_n) \\
&= \sum_{n=1}^{\infty} \big(\pi(C_n \cap A) + \pi(C_n \cap A^{\mathsf{c}})\big) \\
&= \sum_{n=1}^{\infty} \pi(C_n \cap A) + \sum_{n=1}^{\infty} \pi(C_n \cap A) \\
&\geq \mu^*(S \cap A) + \mu^*(S \cap A^{\mathsf{c}}).
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have the desired result.

Now point 2. Fix $A \in \mathcal{A}$. Since $\{A\}$ is a covering of A, we always have $\mu^*(A) \leq \pi(A)$; it remains to prove the opposite inequality.

Again, for any $\epsilon > 0$ there is a covering $\{C_1, C_2, \dots\}$ of A with $\mu^*(A) + \epsilon \geq \sum_{n=1}^{\infty} \pi(C_n)$. Without loss of generality, we can assume the covering is disjoint. (If not, look at the 'disjointification' $C_N \setminus \bigcup_{n=1}^{N-1} C_n$. This only involves complements and finite unions, so is still in $\mathcal{A}$.) Then the sets $C_n \cap A$ are in $\mathcal{A}$ and have disjoint union A. Thus

$$\mu^*(A) + \epsilon \geq \sum_{n=1}^{\infty} \pi(C_n) \geq \sum_{n=1}^{\infty} \pi(C_n \cap A) = \pi(A).$$

Since ϵ was arbitrary we are done. □

7.2 From semialgebras to algebras

We now want to prove the same result for semialgebras. The idea is that we grow the premeasure on a semialgebra $\mathcal{S}$ to a premeasure on the generated algebra $\mathcal{A} = a(\mathcal{S})$, and then apply the previous result.

The following lemma gives us a recipe for building an algebra out of a semialgebra.

Lemma 7.2. *Let X be a nonempty set, and $\mathcal{S}$ be a semialgebra on X. Write*

$$\mathcal{A} = \left\{ \bigcup_{n=1}^{N} A_n : A_n \in \mathcal{S} \text{ disjoint}, N \in \mathbb{N} \right\}$$

for the collection of all finite disjoint unions of sets in $\mathcal{S}$. Then $\mathcal{A}$ is an algebra, and indeed $\mathcal{A} = a(\mathcal{S})$.

Proof. First, that $\varnothing \in \mathcal{A}$ is immediate.

Let's do finite intersections second – finite unions will follow from De Morgan's law once we've done complements. The intersection of two finite disjoint unions is

$$\left(\bigcup_{n=1}^{N} A_n\right) \cap \left(\bigcup_{m=1}^{M} B_n\right) = \bigcup_{n=1}^{N} \bigcup_{m=1}^{M} (A_n \cap B_m)$$

which is also a finite disjoint union of sets in $\mathcal{S}$.

Third, complements. Here we have

$$\left(\bigcup_{n=1}^{N} A_n\right)^{\mathsf{c}} = \bigcap_{n=1}^{N} A_n^{\mathsf{c}}.$$

Since $\mathcal{S}$ is a semialgebra, for each n the complement A_n^{c} is a finite disjoint union of sets in $\mathcal{S}$, so $A_n^{\mathsf{c}} \in \mathcal{A}$. But we've already shown $\mathcal{A}$ is closed under finite intersections, so we are done.

Any algebra containing $\mathcal{S}$ must at least contain the finite disjoint unions in $\mathcal{A}$, so $\mathcal{A}$ is the smallest algebra containing $\mathcal{S}$. □

We now extend the premeasure to the algebra $\mathcal{A}$.

Lemma 7.3. *Let X be a nonempty set, $\mathcal{S}$ be a semialgebra on X, and π a premeasure on $(X, \mathcal{S})$. Then π can be extended to a premeasure $\bar{\pi}$ on $\big(X, a(\mathcal{S})\big)$, and the extension is unique.*

Proof. The lemma above tells us that $a(\mathcal{S})$ consists of all finite disjoint unions from $\mathcal{S}$. Thus we have no choice other than to define

$$\bar{\pi}\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \pi(A_n) \qquad \text{for } A_n \in \mathcal{S} \text{ disjoint.}$$

By Lemma 7.2, this deals with all of $a(\mathcal{S})$. We have to show that $\bar{\pi}$ is well-defined and is a premeasure on $a(\mathcal{S})$.

First well-definition. Suppose $\bigcup_{n=1}^{N} A_n = \bigcup_{m=1}^{M} B_m$, with both sides disjoint unions of sets in $\mathcal{S}$. Then we have

$$A_n = \bigcup_{m=1}^{M} (A_n \cap B_m) \qquad \text{and} \qquad B_m = \bigcup_{n=1}^{N} (A_n \cap B_m),$$

with all unions disjoint, and hence, by finite additivity on $\mathcal{S}$,

$$\pi(A_n) = \sum_{m=1}^{M} \pi(A_n \cap B_m) \qquad \text{and} \qquad \pi(B_m) = \sum_{n=1}^{N} \pi(B_m \cap A_n).$$

Thus we have

$$\bar{\pi}\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \sum_{m=1}^{M} \pi(A_n \cap B_m) = \bar{\pi}\left(\bigcup_{m=1}^{M} B_m\right),$$

so $\bar{\pi}$ is well defined.

That $\bar{\pi}(\varnothing) = \pi(\varnothing) = 0$ is immediate.

Now countable additivity. Suppose $B_1, B_2, \dots$ is a sequence of disjoint sets in $a(\mathcal{S})$ with their union $\bigcup_{n=1}^{\infty} B_n$ also in $a(S)$. By the previous lemma, we can write $B_n = \bigcup_{m=1}^{M_n} A_{nm}$ for disjoint A_{nm} in $\mathcal{S}$. Note that

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{M_n} A_{nm},$$

with both unions disjoint. Then, since π is a premeasure on $\mathcal{S}$, we have

$$\begin{aligned}\bar{\pi}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bar{\pi}\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{M_n} A_{nm}\right) &= \sum_{n=1}^{\infty} \sum_{m=1}^{M_n} \pi(A_{nm}) \\ &= \sum_{n=1}^{\infty} \bar{\pi}\left(\bigcup_{m=1}^{M_n} A_{nm}\right) = \sum_{n=1}^{\infty} \bar{\pi}(B_n).\end{aligned}$$ □

Finally we have:

Theorem 7.4. *Let X be a nonempty set, $\mathcal{A}$ be an algebra on X, and π a premeasure on $(X, \mathcal{A})$. Then there exists a measure μ which extends π to $\sigma(\mathcal{A})$.*

Proof. By Lemma 7.3, the premeasure π extends to a premeasure $\bar{\pi}$ on $a(\mathcal{S})$. Then we can apply Carathéodory's extension theorem. □

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