Lecture 8

π-systems, λ-systems, and the uniqueness lemma

- π-systems and λ-systems
- Dynkin's π–λ theorem
- The uniqueness lemma

8.1 Definitions

Although we could prove the uniqueness part of Carathéodory's extension theorem directly, we can in fact prove a stronger result called 'the uniqueness lemma' by weakening our hypotheses.

An even weaker definition than a semialgebra is a π-system.

Definition 8.1. Let X be a nonempty set, and Π a nonempty collection of subsets of X. Then Π is a *π-system* if

1. Π is closed under finite intersections, in that if $A, B \in \Pi$ then $A \cap B \in \Pi$ also.

That's all!

Clearly every semialgebra is a π-system. Other examples of π-systems are

- the collection of intervals $[a, b)$ including $\varnothing$ (but not the infinite intervals);
- the collection of 'left-infinite' intervals $(-\infty, c]$. (It's easy to show that these generate the Borel σ-algebra.)

Our slogan will be that σ-algebras are often large and unwieldy to deal with, so it's often better to work with a π-system that generates that σ-algebra.

We do need one more set system.

Definition 8.2. Let X be a nonempty set. A collection Λ of subsets of X is called a *λ-system* (or a *d-system* or a *Dynkin system*) on X if it satisfies the following:

1. $\varnothing \in \Lambda$;
2. Λ is closed under complements: for $A \in \Lambda$ we have $A^{\mathsf{c}} \in \Lambda$;
3. Λ is closed under countable disjoint unions: for $A_1, A_2, \dots$ a countably infinite sequence of disjoint sets in Λ, we have $\bigcup_{n=1}^{\infty} A_n \in \Lambda$ also.

Note that this is the same definition as a σ-algebra, except that we only look for closure under unions that are *disjoint*.

It's too easy to be a theorem, but a λ-system is closed under finite disjoint unions (by the standard 'extend with $\varnothing$' argument) and $X = \varnothing^{\mathsf{c}}$ is in every λ-system.

For $\mathcal{C}$ any collection of subsets of X, we can define $\lambda(\mathcal{C})$, the λ-system generated by $\mathcal{C}$, in the usual way.

8.2 Dynkin's π–λ theorem

π-systems and λ-systems work together in interesting ways.

Theorem 8.3. *Let X be a nonempty set, and Σ a collection of subsets of X. Then Σ is a σ-algebra if and only if it is both a π-system and a λ-system.*

Proof. Homework problem. □

Theorem 8.4 (π–λ theorem). *If a λ-system contains a π-system, it also contains the σ-algebra generated by that π-system.*

More precisely, let X be a nonempty set. Suppose Π is a π-system on X, that Λ is a λ-system on X, and that $\Pi \subset \Lambda$. Then $\sigma(\Pi) \subset \Lambda$.

Proof. Consider $\lambda(\Pi)$, the λ-system generated by Π. Clearly it is a λ-system. We claim $\lambda(\Pi)$ is also a π-system. Thus by Theorem 8.3 it is a σ-algebra. But since $\sigma(\Pi)$ is the smallest σ-algebra containing Π, since every σ-algebra is a λ-system, and since Λ is λ-system containing Π we have the chain of inclusions

$$\Pi \subset \sigma(\Pi) = \lambda(\Pi) \subset \Lambda.$$

This proves the theorem.

It remains to prove the claim that $\lambda(\Pi)$ is a π-system. We need to show that for $A, B \in \lambda(\Pi)$ we also have $A \cap B \in \lambda(\Pi)$.

For step one, fix $A \in \lambda(\Pi)$, and write

$$\mathcal{L}_A = \big\{B \subset X : A \cap B \in \lambda(\Pi)\big\}$$

for the set of B such that $A \cap B$ is indeed in $\lambda(\Pi)$. We claim that $\mathcal{L}_A$ is a λ-system.

That $\varnothing \in \mathcal{L}_A$ is immediate. It's also clear that $A^{\mathsf{c}} \in \mathcal{L}_A$, since $A \cap A^{\mathsf{c}} = X$, and X is every λ-system, so certainly in $\lambda(\Pi)$. We also see that if $\bigcup_{n=1}^{\infty} B_n$ is a disjoint union, then

$$A \cap \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} (A \cap B_n)$$

is a disjoint union also, so $\mathcal{L}_A$ is closed under countable disjoint unions. Thus $\mathcal{L}_A$ is indeed a λ-system.

For step two, suppose that in fact $A \in \Pi$. Then for any $B \in \Pi$ we clearly have, by the definition of a π-system, that $A \cap B \in \Pi \subset \lambda(\Pi)$. So $\Pi \subset \mathcal{L}_A$. So since $\mathcal{L}_A$ is a λ-system containing Π, then we certainly have $\lambda(\Pi) \subset \mathcal{L}_A$. From the definition of $\mathcal{L}_A$, we see that we have that $A \cap B \in \lambda(\Pi)$ whenever $A \in \Pi$ and $B \in \lambda(\Pi)$.

We're halfway there. Now for step three, we swap the roles of A and B.

Fix $B \in \lambda(\Pi)$, and set

$$\mathcal{L}_B = \big\{A \subset X : A \cap B \in \lambda(\Pi)\big\}.$$

As before, this is a λ-system. But what we showed in step two was that $\Pi \subset \mathcal{L}_B$. Hence $\lambda(\Pi) \subset \mathcal{L}_B$ for any $B \in \lambda(\Pi)$.

But this is precisely the statement that for any $A, B \in \lambda(\Pi)$ we also have $A \cap B \in \lambda(\Pi)$. □

8.3 Proof of the uniqueness lemma

Here we prove a result that is more general than the uniqueness part of Carathéodory's extension theorem.

Theorem 8.5 (The uniqueness lemma)**.** *Let (X, Σ, μ) be a measure space with $\Sigma = \sigma(\Pi)$ for some π-system Π, and where μ is σ-finite on Π, in that X can be written as a countable union of disjoint sets in Π of finite measure. Suppose further that there is another measure ν on (X, Σ) that is equal to μ on Π, in that $\mu(A) = \nu(A)$ for $A \in \Pi$. Then μ and ν are equal on all of Σ.*

In other words, if two measures agree on a π-system, they agree on the σ-algebra generated by that π-system. (Subject to the σ-finiteness condition.)

Note that this result does not guarantee that any particular such μ exists, just that if there is such a μ, it's the only one.

The result is easy when we have that $\mu(X) = \nu(X)$ and the common value is finite, so let's do that first. This simple version is sufficient for proving uniqueness of probability measures, for example.

Proof when $\mu(X) = \nu(X) < \infty$. Write

$$\Lambda = \big\{A \in \Sigma : \mu(A) = \nu(A)\big\}$$

for the collection of sets where μ and μ agree. By hypothesis, $\Pi \subset \Lambda$. The key point is that Λ is in fact a λ-system. (Proving this is a homework problem.) Then by the π–λ theorem, we have that $\sigma(\Pi) \subset \Lambda$, so μ and ν agree on the whole of $\Sigma = \sigma(\Pi)$. □

Now for the more general case.

Proof of general case. Let $B_1, B_2, \dots$ be the disjoint sequence in Π each with finite measure whose union is X. Then for each B_n, we have that $\mu(B_n) = \nu(B_n)$.

Along the lines of the previous proof, write

$$\Lambda_n = \big\{A \in \Sigma : \mu(A \cap B_n) = \nu(A \cap B_n)\big\}.$$

As before, this is λ-system (check, if you must) with $\Pi \subset \Lambda_n$, so in fact $\Sigma \subset \Lambda_n$. Hence $\mu(A \cap B_n) = \nu(A \cap B_n)$ for all $A \in \Sigma$.

For any $A \in \Sigma$, we have $A = \bigcup_{n=1}^\infty (A \cap B_n)$ with the union disjoint. Hence

$$\mu(A) = \sum_{n=1}^\infty \mu(A \cap B_n) = \sum_{n=1}^\infty \nu(A \cap B_n) = \nu(A). \qquad \square$$

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