`MA40042` Measure Theory and Integration

Problem Sheet 1

Solutions should be submitted to the `MA40042` pigeonhole by 1700 on ***Monday 10 October****.*

Work will be returned and answers discussed in the problems class on Tuesday 11 October.

1. Let X be a nonempty set. Show that the following are σ-algebras on X:
   (a) the powerset of X;
   (b) the trivial σ-algebra;
   (c) $\{\varnothing, A, A^{\mathsf{c}}, X\}$ for some $A \subset X$;
   (d) $\{A \subset X : A \text{ is countable or co-countable}\}$.

2. Let (X, Σ) be a measurable space.
   (a) Show that the following sets are in Σ:
      i. X;
      ii. $\bigcup_{n=1}^{N} A_n$, where $A_1, A_2, \dots, A_N$ is a finite sequence of sets in Σ;
      iii. $\bigcap_{n=1}^{\infty} A_n$, where $A_1, A_2, \dots$ is a countably infinite sequence of sets in Σ;
      iv. $\bigcap_{n=1}^{N} A_n$, where $A_1, A_2, \dots, A_N$ is a finite sequence of sets in Σ (give two proofs if you can);
      v. $B \setminus A$, where $A, B \in \Sigma$ with $A \subset B$;
      vi. $A \triangle B$, where $A, B \in \Sigma$.
   (b) Let $A_1, A_2, \dots$ be a countably infinite sequence of sets in Σ.
      i. Write $A_{\text{i.o.}}$ for the set of points $x \in X$ in infinitely many of the A_ns. (In probability settings, this is the event that the events A_n occur 'infinitely often'.) Show that $A_{\text{i.o.}} \in \Sigma$.
      ii. Write A_{ev} for the set of points $x \in X$ in every A_n for n sufficiently large. (In probability settings, this is the event that the events A_n occur 'eventually' – although 'eventually always' might be a better term.) Show that $A_{\text{ev}} \in \Sigma$. (Give two proofs if you can.)

3. (a) Show that every algebra on a finite set X is a σ-algebra.
   (b) Let $\mathcal{C}$ be a collection of subsets of a nonempty set X.
      i. Give the 'obvious' definition of $a(\mathcal{C})$, the algebra generated by $\mathcal{C}$.
      ii. Briefly show that $a(\mathcal{C})$ is indeed an algebra on X.
   (c) Let X be a nonempty set, and let
   $$\mathcal{C} = \big\{\{x\} : x \in X\big\}$$
   be the collection of singletons in X. Explain what $\sigma(\mathcal{C})$ and $a(\mathcal{C})$ are when X is
      i. finite,
      ii. countable infinite,
      iii. uncountably infinite.
   (You may find the word 'co-finite' useful.)
   (d) Hence, give an concrete example of a set X and an algebra on X that is not a σ-algebra. Give an explicit counterexample to it being a σ-algebra.

4. (a) We work in $\mathbb{R}^d$.
      i. Show that a union of open sets is open.
      ii. Show that an intersection of closed sets is closed.
   (b) Let $\mathcal{K}$ denote the collection of compact sets in $\mathbb{R}$. Show that $\sigma(\mathcal{K}) = \mathcal{B}(\mathbb{R}^d)$.

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