Solutions: Sheet 3, Questions 2 and 3

2. Our aim is to construct a probability space representing an infinite sequence of coin tosses. Let $\Omega = \{\mathtt{H},\mathtt{T}\}^{\mathbb{N}}$, so $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots) \in \Omega$ represents an infinite string of heads and tails. Given $\mathbf{x} \in \{\mathtt{H},\mathtt{T}\}^n$ for some n, write $C(\mathbf{x})$ for the *cylinder set*

$$C(\mathbf{x}) = \{\boldsymbol{\omega} \in \Omega : \omega_1 = x_1, \omega_2 = x_2, \dots, \omega_n = x_n\}.$$

Write $\mathcal{C}$ for the empty set and all cylinder sets $C(\mathbf{x})$ for $\mathbf{x} \in \{\mathtt{H},\mathtt{T}\}^n$ for any $n \in \mathbb{N}$.

(a) Show that $\mathcal{C}$ is a semialgebra on Ω

Solution: We have three things to check.

1. That $\varnothing \in \mathcal{C}$ is immediate.
2. Intersections with the empty set are empty, so consider $C(\mathbf{x}) \cap C(\mathbf{y})$. This is empty if for some i we have $x_i \neq y_i$, so it's only nonempty is $\mathbf{x}$ is a prefix of $\mathbf{y}$, (or vice versa, but without loss of generality let's say $\mathbf{x}$ is the prefix). In that situation, $C(\mathbf{x}) \cap C(\mathbf{y}) = C(\mathbf{y})$. For example, $C(\mathtt{H}) \cap C(\mathtt{HTT}) = C(\mathtt{HTT})$.
3. Suppose $\mathbf{x} \in \{\mathtt{H},\mathtt{T}\}^n$. Then we have

$$C(\mathbf{x})^{\mathsf{c}} = \bigcup_{\substack{\mathbf{y} \in \{\mathtt{H},\mathtt{T}\}^n \\ \mathbf{y} \neq \mathbf{x}}} C(\mathbf{y}),$$

with the union finite and disjoint.

Define $\pi \colon \mathcal{C} \to [0,\infty]$ by $\pi(\varnothing) = 0$ and for $\mathbf{x} \in \{\mathtt{H},\mathtt{T}\}^n$ put $\pi\big(C(\mathbf{x})\big) = 2^{-n}$.

(b) Show that π is finitely additive on disjoint sets in $\mathcal{C}$.

Solution: If we have a cylinder set $C(\mathbf{x})$ with $\mathbf{x}$ of length n, we can instead deal with strings of length $n+k$ by writing

$$C(\mathbf{x}) = \bigcup_{\mathbf{y} \in \{\mathtt{H},\mathtt{T}\}^k} C(\mathbf{x},\mathbf{y}),$$

where $(\mathbf{x},\mathbf{y})$ denotes the concatenation of $\mathbf{x}$ and $\mathbf{y}$. Note that the union is disjoint and finitely additive, in that the premeasures are

$$2^k \pi\big(C(\mathbf{x},\mathbf{y})\big) = 2^k 2^{-(n+k)} = 2^{-n}.$$

Suppose then we have a finite disjoint union $C(\mathbf{z}) = \bigcup_{n=1}^N C(\mathbf{x}_n)$ with $\mathbf{z}$ of length m. Note that each $\mathbf{x}_n$ has $\mathbf{z}$ as a prefix. Let $m + j_n$ be the length of the string $\mathbf{x}_n$, and $m+j$ be longest length of any string. Then use the above to rewrite each of $C(\mathbf{x}_n)$s in terms of strings of length $m+j$. By the above this preserves the premeasure. Then we have

$$C(\mathbf{z}) = \bigcup_{n=1}^N \bigcup_{\mathbf{y} \in \{\mathtt{H},\mathtt{T}\}^{j-j_n}} C(\mathbf{x}_n,\mathbf{y}) = \bigcup_{\mathbf{w} \in \{\mathtt{H},\mathtt{T}\}^{j-m}} C(\mathbf{z},\mathbf{w}).$$

Thus the premeasure is additive as above.

(c) Show that no cylinder set can be written as a countably infinite disjoint union of cylinder sets. *(This is quite hard. If you can't give a proof, try to sketch the general idea, or even just explain why one might expect this to be true.)*

Solution: The idea is that $\boldsymbol{\omega} \in C(\mathbf{x})$ is a condition only on the first n coinflips of $\boldsymbol{\omega}$ for some n, whereas whether $\boldsymbol{\omega}$ is in a countably infinite disjoint union could not be determined by any finite prefix of $\boldsymbol{\omega}$.

Assume, seeking a contradiction, that $C(\mathbf{x}) = \bigcup_{n=1}^\infty C(\mathbf{x}^{(n)})$, with the union a disjoint one. Thus the sets $B_N = C(\mathbf{x}) \setminus \bigcup_{n=1}^N C(\mathbf{x}^{(n)})$ are nonempty, with $B_1 \supset B_2 \supset \cdots$. Further whether $\boldsymbol{\omega} \in B_N$ is determined by a finite prefix of $\boldsymbol{\omega}$. Then there exists a z_1, either $\mathtt{H}$ or $\mathtt{T}$, such that every B_N contains a string starting $\omega_1 = z_1$. Fix such a z_1. But by repeating, there exists a z_2 such that every B_N contains a string starting $(\omega_1,\omega_2) = (z_1,z_2)$. This way, we can produce an arbitrarily long string prefix that features in every B_N. But since whether $\boldsymbol{\omega} \in B_N$ can be decided by a finite prefix, by going beyond that finite number, we see that $\bigcap_{n=1}^\infty B_n$ is nonempty. Thus $C(\mathbf{x}) \setminus \bigcup_{n=1}^\infty C(\mathbf{x}^{(n)})$ is nonempty, and we have out contradiction.

(d) Deduce that π is a premeasure on $\mathcal{C}$.

> **Solution:** That $\pi(\varnothing) = 0$ is immediate.
> By (c), any countable disjoint union in $\mathcal{C}$ can only include finitely many nonempty sets (perhaps with repetition), so finite additivity proves π is a premeasure.

Hence, by Carathéodory's extension theorem, π can be extended to a measure $\mathbb{P}$ on $(\Omega, \sigma(\mathcal{C}))$.

(e) Show that any such $\mathbb{P}$ is a probability measure.

> **Solution:** Clearly
> $$\mathbb{P}(\Omega) = \mathbb{P}\big(C(\mathtt{H})\big) + \mathbb{P}\big(C(\mathtt{T})\big) = \pi\big(C(\mathtt{H})\big) + \pi\big(C(\mathtt{T})\big) = 1/2 + 1/2 = 1.$$

(f) Show that $\mathbb{P}$ is the unique extension of π.

> **Solution:** Carathéodory's extension theorem requires that the measure $\mathbb{P}$ is σ-finite, which it is, since $\mathbb{P}(\Omega) = 1$.

3. Let (X, Σ, μ) and (Y, Π, ν) be two measure spaces. Write

$$\mathcal{S} = \{A \times B : A \in \Sigma, B \in \Pi\}$$

and put $\Sigma \otimes \Pi = \sigma(\mathcal{S})$ for the σ-algebra on $X \times Y$ generated by $\mathcal{S}$.

(a) Suppose X and Y are countable. What is $\mathcal{P}(X) \otimes \mathcal{P}(Y)$?

> **Solution:** $\mathcal{P}(X) \otimes \mathcal{P}(Y) = \mathcal{P}(X \times Y)$

(b) Suppose $X = Y = \mathbb{R}$. Show that $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$.

> **Solution:** First let's show $\mathcal{B}(\mathbb{R}^2) \subset \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$, for which it suffices to show that $\mathcal{I}_2 \subset \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$. This is obviously true.
> Second we need to show $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \subset \mathcal{B}(\mathbb{R}^2)$. Set
> $$\mathcal{C} := \big\{A \in \mathcal{B}(\mathbb{R}) : A \times \mathbb{R} \in B(\mathbb{R}^2)\big\}.$$
> It's easy to check that $\mathcal{C}$ is a σ-algebra, and also that it contains all one-dimensional open sets. Hence $\mathcal{C} \subset \mathcal{B}(\mathbb{R}^2)$. Similarly, we can set
> $$\mathcal{D} := \big\{B \in \mathcal{B}(\mathbb{R}) : \mathbb{R} \times B \in B(\mathbb{R}^2)\big\},$$
> and see that $\mathcal{D}$ is a σ-algebra with $\mathcal{D} \subset B(\mathbb{R}^2)$. But $\mathcal{C} \cap \mathcal{D} = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$, so $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \subset \mathcal{B}(\mathbb{R}^2)$, as desired.

(c) Show, in general, that $\mathcal{S}$ is a semialgebra on $X \times Y$.

> **Solution:**
> 1. $\varnothing = \varnothing \times \varnothing$.
> 2. $(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$.
> 3. $(A \times B)^{\mathsf{c}} = (A^{\mathsf{c}} \times B) \cup (A \times B^{\mathsf{c}}) \cup (A^{\mathsf{c}} \times B^{\mathsf{c}})$.

Write $\pi(A \times B) = \mu(A)\nu(B)$. You may assume (and will probably prove later in the course) that π is a premeasure on $(X, \mathcal{S})$. Hence π extends to a measure on $\Sigma \otimes \Pi$, which is called the *product measure* and written $\mu \times \nu$.

(d) Show that if μ and ν are both σ-finite, then the product measure is unique.

> **Solution:** We need to show that the product measure $\mu \times \nu$ is σ-finite.
> By assumption, there exist $A_1, A_2, \ldots$ in Σ with union X and measures $\mu(A_n) < \infty$, and $B_1, B_2, \ldots$ in Π with union Y and measures $\nu(B_m) < \infty$. Then we have
> $$\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (A_n \times B_m) = X \times Y,$$
> where the union is countable and
> $$(\mu \times \nu)(A_n \times B_m) = \mu(A_n)\nu(B_m) < \infty.$$

(e) Let X and Y be countable, and endowed with their powersets and counting measures. What is the corresponding product measure.

> **Solution:** Counting measure on $X \times Y$.

(f) Outline (without proofs) an alternative construction of the Lebesgue measure on $\mathbb{R}^d$ for $d \geq 2$.

> **Solution:** Define the Lebesgue measure λ on $(\mathbb{R}, \mathcal{B})$ as in lectures. Then inductively define the Lebesgue measure λ_{d+1} on
> $$\big(\mathbb{R}^d \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}\big) = \big(\mathbb{R}^{d+1}, \mathcal{B}(\mathbb{R}^{d+1})\big)$$
> as the produce measure $\lambda_d \times \lambda$.