Topics in Discrete Mathematics
Part 2: Introduction to Graph Theory

Past Exam Question: Solutions

You are strongly encouraged to attempt the question first without looking at the solutions.

The following was Question 1 in the May–June 2013 exam for Topics in Discrete Mathematics. The exam contained three questions, of which two had to be answered in 1 hour and 30 minutes. So you should be trying to do this question in 35 or 40 minutes.

1. (a) Let $G = (V, E)$ be a graph. Define

i. (2 marks) a walk,

Solution: A *walk* is a sequence of vertices $v_0v_1v_2 \cdots v_k$ such that $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ are all edges.

ii. (2 marks) a trail,

Solution: A *trail* is a walk $v_0v_1v_2 \cdots v_k$ such that the edges $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ are distinct.

iii. (2 marks) a cycle,

Solution: A *cycle* is a walk $v_0v_1v_2 \cdots v_k$ where $v_0 = v_k$, but all other vertices are distinct.

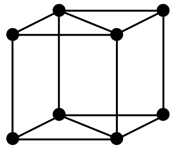
iv. (2 marks) a circuit,

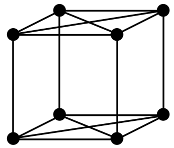
Solution: A *circuit* is a trail $v_0v_1v_2 \cdots v_k$ where $v_0 = v_k$.

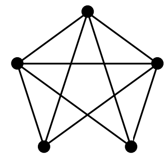
v. (2 marks) an Eulerian circuit.

Solution: An *Eulerian circuit* is a circuit that uses every edge.

(b) (10 marks) For each of the following four graphs, decide if it has an Eulerain trail[1], an Eulerian circuit, both, or neither. Give a brief justification in each case.

i. 

ii. 

iii. 

iv. The complete graph K_n for $n \geq 4$ even.

Solution: Since all the graphs are connected, we need only check the parity of degrees.

i. There are four vertices of odd degree, so there is neither an Eulerian trail nor an Eulerian circuit.

ii. All vertices have even degree, so there is an Eulerian circuit, which is also an Eulerian trail.

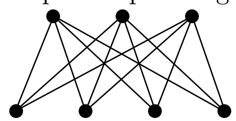
iii. There are two vertices of odd degree, so there is an Eulerian trail but no Eulerian circuit.

iv. Each vertex has degree $n - 1$, which is odd. Hence there is neither an Eulerian trail nor an Eulerian circuit.

(c) i. (2 marks) Define what it means for a graph to have an Hamiltonian cycle.

Solution: A *Hamiltonian cycle* is a cycle that visits every vertex.

ii. (6 marks) Consider the complete bipartite graph $K_{3,4}$.



Prove that this graph does not have a Hamiltonian cycle.

Solution: Suppose there exists a Hamiltonian cycle. Since it visits every vertex exactly once, it is necessarily of length 7. Without loss of generality, say it starts on the bottom of the bipartition. Then it goes bottom-top-bottom-top and so on, and so the path consisting of the first 6 steps ends on the bottom (as 6 is even). The final edge must lead from this vertex, on the bottom, back to

[1] An *Eulerian trail* is a trail that uses every edge. For more, see Question 5. (b) on the Problem Sheet.

the starting vertex, also on the bottom, which is impossible. Hence no Hamiltonian cycle exists.

(d) (8 marks) Let $G = (V, E)$ be a connected bipartite graph with $|V| = n$ and $|E| = m \geq 2$. Prove that if G is planar, then $m \leq 2n - 4$.

Solution: Note that a bipartite graph contains no 3-cycles (for the same parity reasons as above). Hence each face has degree $d(F) \geq 4$ (as we have ruled out the case of a single face with degree less than four by demanding $m \geq 2$).
Then by the handshake lemma for faces, we have

$$2m = \sum_{\text{faces } F} d(F) \geq \sum_{\text{faces } F} 4 = 4f,$$

where f is the number of faces.
Euler's formula tells us that, for a connected graph, $n - m + f = 2$. Substituting this into the above, we get

$$2m \geq 4(2 - n + m) = 8 - 4n + 4m.$$

Rearranging gives $4n - 8 \geq 2m$, and dividing by 2 gives the desired result.

(e) i. (2 marks) Define the adjacency matrix of an n-vertex graph $G = (V, E)$.

Solution: The *adjacency matrix* is the $n \times n$ matrix $\mathsf{A} = (a_{ij} : i, j \in V)$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E, \\ 0 & \text{if } ij \notin E. \end{cases}$$

ii. (2 marks) For a graph $G = (V, E)$, we define the complement $\overline{G} = (V, \overline{E})$, where $e \in \overline{E}$ if and only if $e \notin E$. Let G be a graph with n vertices, and let A be its adjacency matrix. Write down the adjacency matrix $\overline{\mathsf{A}}$ of its complement in terms of A and M_n, where M_n is the adjacency matrix of the complete graph K_n.

Solution: $\overline{\mathsf{A}} = \mathsf{M}_n - \mathsf{A}$.

iii. (2 marks) Define what it means for two graphs G_1, G_2 to be isomorphic.

Solution: The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijection $\phi \colon V_1 \to V_2$ such that $\phi(u)\phi(v) \in E_2$ if and only if $uv \in E_1$.

iv. (8 marks) We say that G is self-complementary if G is isomorphic to $\overline{G}$. Prove that if G is self-complementary, then $n \equiv 0$ or $1 \pmod 4$.

Solution: From iii., we see that G and $\overline{G}$ must have the same number of edges, and from ii., we see that the sum of their numbers of edges must be the number of edges of the complete graph. Hence, G and $\overline{G}$ each have

$$\frac{1}{2}\binom{n}{2} = \frac{1}{2}\frac{n(n-1)}{2} = \frac{n(n-1)}{4}$$

edges.
Clearly this number of edges must be an integer. So $n(n-1)$ must be divisible by 4. Since n and $(n-1)$ cannot both be divisible by 2, we must have that either n or $n-1$ is divisible by 4. This is precisely the condition that $n \equiv 0$ or $1 \pmod 4$.

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