Topics in Discrete Mathematics
Part 2: Introduction to Graph Theory

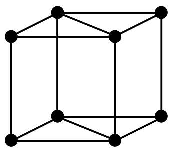
Past Exam Question

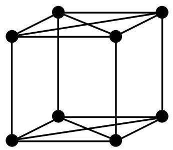
Solutions are available online at
`http://www.maths.bris.ac.uk/~mampa/teaching/`
although you are strongly encouraged to attempt the question first without looking at the solutions.

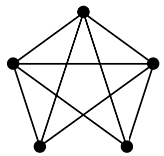
The following was Question 1 in the May–June 2013 exam for Topics in Discrete Mathematics. The exam contained three questions, of which two had to be answered in 1 hour and 30 minutes. So you should be trying to do this question in 35 or 40 minutes.

1. (a) Let $G = (V, E)$ be a graph. Define
   i. (2 marks) a walk,
   ii. (2 marks) a trail,
   iii. (2 marks) a cycle,
   iv. (2 marks) a circuit,
   v. (2 marks) an Eulerian circuit.

   (b) (10 marks) For each of the following four graphs, decide if it has an Eulerain trail[1], an Eulerian circuit, both, or neither. Give a brief justification in each case.

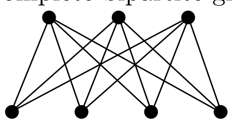
   i. 

   ii. 

   iii. 

   iv. The complete graph K_n for $n \geq 4$ even.

   (c) i. (2 marks) Define what it means for a graph to have an Hamiltonian cycle.
   ii. (6 marks) Consider the complete bipartite graph $K_{3,4}$.

   Prove that this graph does not have a Hamiltonian cycle.

[1] An *Eulerian trail* is a trail that uses every edge. For more, see Question 5. (b) on the Problem Sheet.

   (d) (8 marks) Let $G = (V, E)$ be a connected bipartite graph with $|V| = n$ and $|E| = m \geq 2$. Prove that if G is planar, then $m \leq 2n - 4$.

   (e) i. (2 marks) Define the adjacency matrix of an n-vertex graph $G = (V, E)$.
   ii. (2 marks) For a graph $G = (V, E)$, we define the complement $\overline{G} = (V, \overline{E})$, where $e \in \overline{E}$ if and only if $e \notin E$. Let G be a graph with n vertices, and let A be its adjacency matrix. Write down the adjacency matrix $\overline{\mathsf{A}}$ of its complement in terms of A and M_n, wher M_n is the adjacency matrix of the complete graph K_n.
   iii. (2 marks) Define what it means for two graphs G_1, G_2 to be isomorphic.
   iv. (8 marks) We say that G is self-complementary if G is isomorphic to $\overline{G}$. Prove that if G is self-complementary, then $n \equiv 0$ or $1 \pmod 4$.

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