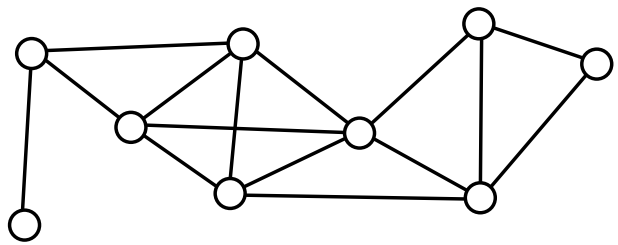
Topics in Discrete Mathematics
Part 2: Introduction to Graph Theory

Lecture 1
Graphs

- Definition of a graph
- Common examples: K_n, P_n, C_n
- Degrees and the handshake lemma
- Graph isomorphism
- Subgraphs and unions

1.1 Definitions and examples

Graphs are things that look like this:



We have a some blobs, called *vertices*, joined by lines, called *edges*.

We can think of the vertices as objects, and the edges representing relations between pairs of objects. For example, the vertices might represent towns and the edges roads between them. Or the vertices might represent computers and the edges wired connections between them.

But these collections of blobs and lines can fascinating as objects of mathematics interest in their own right.

We will define edges by the two vertices at their ends. This gives the following definition, the most important of the course:

Definition 1.1. A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* and a set

$$E \subseteq V^{(2)} := \big\{\{u, v\} : u, v \in V, u \neq v\big\}$$

of unordered pairs from V called *edges*.

If $u, v \in V$ are vertices and $e = \{u, v\} \in E$ is an edge between them, we say that u and v are *adjacent*, and that e is *incident* to u and v.

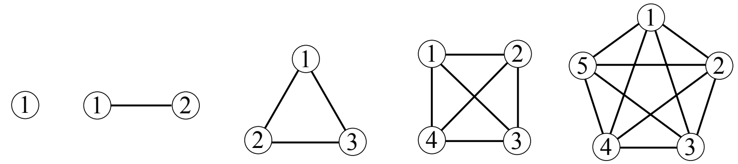
The number of vertices $n = |V|$ is called the *order* of G, and the number of edges $m = |E|$ is called the *size* of G.

Throughout we will abbreviate an edge $\{u, v\}$ as just uv. Remember that these are unordered pairs, so uv and vu denote the same edge.

Here are some common examples of graphs with n vertices.

Example 1.2. The *complete graph* K_n on n vertices is defined by

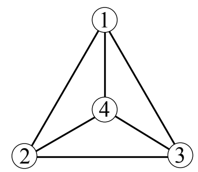
$$V = [n] = \{1, 2, \dots, n\}, \qquad E = V^{(2)} = [n]^{(2)} = \{12, 13, 14, \dots, (n-1)n\}.$$



K_1 K_2 K_3 K_4 K_5

We call K_1, with one vertex and no edges the *trivial* graph. Some theorems in this course apply to *nontrivial* graphs – that is, graphs with two or more vertices.

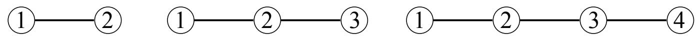
Note that we can also draw K_4 as



where none of the edges cross each other. In Lecture 4 we'll talk more about graphs that can be drawn without edge crossings.

Example 1.3. The *path* P_n on $n \geq 2$ vertices (so of length $n - 1$) is defined by

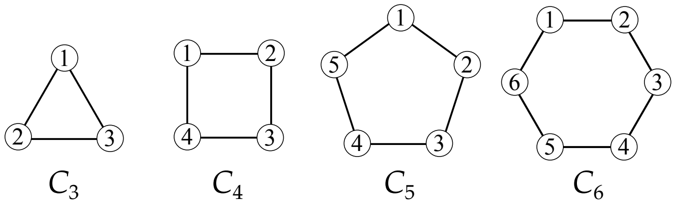
$$V = [n], \qquad E = \{12, 23, 34, \dots, (n-1)n\}.$$



P_2 P_3 P_4

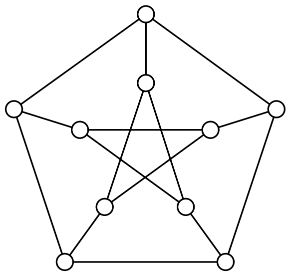
Example 1.4. The *cycle* C_n on $n \geq 3$ vertices is defined by

$$V = [n], \qquad E = \{12, 23, 34, \dots, (n-1)n, n1\}.$$



Note that $P_2 = K_2$ and $C_3 = K_3$ – the latter graph is often called the *triangle*.

Example 1.5. A graph that will pop up a few times in this course is the *Petersen graph* shown below:



For reasons will see in a moment, we haven't labelled the vertices. But note that if we had, the graph would be unambiguously defined by the picture – so we'll often not write down V and E explicitly for graphs we consider.

Looking carefully at Definition 1.1, it's worth emphasising a few points:

- The edge set E is a set, so can include each pair uv at most once. So we are not allowed multiple edges between vertices.

- Edges uv have to be between distinct vertices. So we can't have a 'loop' from a vertex to itself.

Note that some textbooks differ from our convention here, and refer to our graphs – no multiple edges, no loops – as 'simple' graphs.

- Edges uv are unordered pairs. So an edge $uv = vu$ goes between vertices u and v an are never 'directed' from one vertex to another.

Directed graphs are interesting in their own right, but are outside the scope of this course (except for the final question on the Problem Sheet).

- Edges uv consist of pairs of vertices. So we don't have 'hyperedges' between three or more vertices.

Again, 'hypergraphs' with these hyperedges are interesting but outside our scope.

- Although our definition allows V to be infinite, from now on we always assume (unless we say otherwise) that our graphs have a finite number of vertices.

Keen students can consider it an exercise to work out which theorems in this coursealso hold when V is countable or uncountably infinite.

Definition 1.6. Consider a graph $G = (V, E)$. The *degree* $d(v)$ of a vertex $v \in V$ is defined as the number of edges incident to v:

$$d(v) := |\{e \in E : v \in e\}|.$$

Equivalently, $d(v)$ is the number of vertices adjacent to v

$$d(v) = |\{u \in V : vu \in E\}|.$$

The seqence of degrees $(d(v) : v \in V)$, usually written in decreasing order, is called the *degree sequence* of G.

We write $\Delta := \max_v d(v)$ for the maximum degree, and $\delta := \min_v d(v)$ for the minimum degree.

So, for example, the degree sequence of K_4 is $(3, 3, 3, 3)$ and the degree sequence of P_5 is $(2, 2, 2, 1, 1)$.

Let's make the first theorem of the course an easy one.

Theorem 1.7 (Handshake lemma)**.** *The sum of the degrees is twice the number of edges. That is, for a graph $G = (V, E)$, we have*

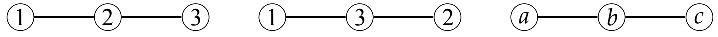
$$\sum_{v \in V} d(v) = 2|E|.$$

In particular, the sum of the degrees is always even.

Proof. Go through the vertices of the graph counting the edges incident of each vertex. This is clearly $\sum_v d(v)$. We've now counted each edge exactly twice, once for each vertex it's incident to, which is $2|E|$. $\square$

1.2 Graph isomorphism

The following graphs are obviously, in some sense, the same graph.



G_1 G_2 G_3

However, G_1 and G_3 have different vertex sets, and all three have different edge sets. But the point is that if we relabel the vertices of G_2 and G_3, then they can be made equal.

If graphs can be relabelled to be made equal, then we say they are *isomorphic*. Throughout this course we will consider isomorphic graphs to the same graph (so in particular, we often won't bother to label the vertices of graphs in pictures).

Definition 1.8. Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there exists a bijection $\phi\colon V \to V'$ such that $uv \in E$ is an edge in G if and only if $\phi(u)\phi(v) \in E'$ is an edge in G'.

So we can show that G_1 and G_3 above are isomorphic by writing down the isomorphism

$$\phi(1) = a, \qquad \phi(2) = b, \qquad \phi(3) = c.$$

1.3 New graphs from old

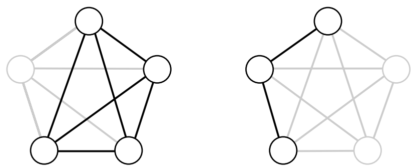
To concepts it will be useful to have are that of the subgraph and of graph unions.

Definition 1.9. A graph $G' = (V', E')$ is a *subgraph* of another graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E \cap V'^{(2)}$.

In other words, to get to a subgraph, we can first delete some vertices, we then must delete all edges incident to those vertices, and then we can further delete more edges.

As ever, we will consider a graph H to be a subset of a graph G if G contains a subgraph isomorphic to H.

So, for example, K_4 and P_3 are both subgraphs of K_5.

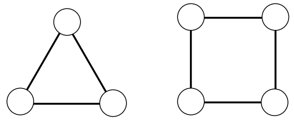


K_4 P_3

The graph union is what you get by simply drawing to graphs next to each other.

Definition 1.10. Consider two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets. The *graph union* is defined by $G_1 \cup G_2 := (V_1 \cup V_2, E_1 \cup E_2)$.

Again, if the vertex sets are not disjoint, we will replace one of the graphs by an isomorphic copy such that the vertex sets are disjoint. Hence the single graph shown below is $K_3 \cup C_4$.



Next time: *Taking walks around graphs: paths, trails, cycles and circuits.*

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