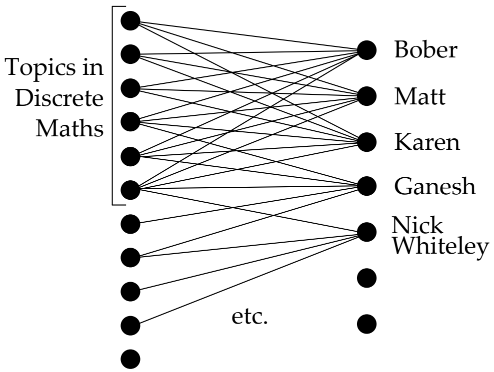
Topics in Discrete Mathematics
Part 2: Introduction to Graph Theory

Lecture 3
Bipartite Graphs

- Definition of bipartite graphs
- Complete bipartite graphs
- Matchings and Hall's marriage theorem

3.1 Basics

Consider the following graph: the vertices will be the people in the Bristol maths department, and there will be an edge between two people if one has lectured the other this year.



Note that the vertices seem to naturally split into two categories: the lecturers, and the students. All edges in the graph have one end in the set of students and one in the set of lecturers – there are no edges within the set of students, and no edges within the set of lecturers.

Graphs that can be partitioned this way are called *bipartite*.

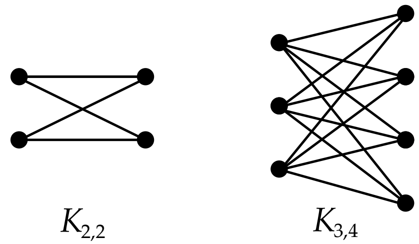
Definition 3.1. A graph $G = (V, E)$ is *bipartite* if we can write V as a disjoint union $V = X \cup Y$, with X and Y nonempty, such that $X^{(2)} \cap E = Y^{(2)} \cap E = \varnothing$.

Bipartite graphs arise naturally when we have two different types of vertices. For example, let X be the set of lecture classes set for 3pm on a Thursday, let Y be the set of lecture rooms, with an edge xy if class x will fit into room y. This is clearly bipartite.

A useful graph to work with is the complete bipartite graph.

Example 3.2. The complete bipartite graph $K_{a,b}$ with a vertices on the left and b vertices on the right is defined by

$$X = [a] \qquad Y = [b] \qquad E = \{xy : x \in [a], y \in [b]\}.$$



(Technically we've given some vertices on the left and right the same label, but using the 'isomorphism trick' again, we won't let this bother us.)

Note that $K_{a,b}$ is isomorphic to $K_{b,a}$.

Other examples of bipartite graphs include the even cycles, C_n for odd n, and the paths P_n.

3.2 Matchings

A common toy example of a bipartite graph is this: Let X be a set of girls, let Y be a set of boys, and let xy be an edge if girl x knows boy y. We might then wonder if we can manage to marry off each girl to one of the boys she knows.

Definition 3.3. Let $G = (X \cup Y, E)$ be a bipartite graph. A *matching* ϕ on G is an injection $\phi\colon X \to Y$ such that, for each $x \in X$, $x\phi(x) \in E$ is an edge.

Note that ϕ being an injection ensures that each girl gets married to a different boy (no polygamy!).

The example of marrying boys to girls is a bit silly, in assuming that people are willing to be marrying anyone they know (not to mention somewhat homophobic). However, there are cases where matchings can be useful. For example, in our lecture rooms and lecture classes above, a matching would be a way of scheduling each class into room it will fit in.

Whether or not a bipartite graph has a matching will depend on the size of certain *neighbourhoods*.

Definition 3.4. Consider a graph $G = (V, E)$. The *neighbourhood* $N(U)$ of a subset of vertices $U \subseteq V$ is the set

$$N(U) := \{v \in V : uv \in E \text{ for some } u \in U\}$$

of vertices adjacent to some vertex in U.

So if $A \subseteq X$ is a subset of girls, $N(A) \subseteq Y$ is the set of boys they collectively know.

Suppose we have a set A of $|A| = i$ girls. If $|N(A)| < i$, then these i girls between them know fewer than i boys. Hence, it clearly won't be possible to marry each of these girls to a different boy. Hence, for a matching to exist, we clearly must have

$$|N(A)| \geq |A| \qquad \text{for all } A \subseteq X.$$

This is known as *Hall's condition.*

Perhaps surprisingly, Hall's condition is not only necessary but also sufficient for a matching to exist.

Theorem 3.5 (Hall's marriage theorem)**.** *Let $G = (X \cup Y, E)$ be a bipartite graph. Then a matching exists if and only if Hall's condition holds, in that*

$$|N(A)| \geq |A| \qquad \textit{for all } A \subseteq X.$$

Proof. (*Only if*) Explained above.

(*If*) We work by induction on the number of girls $|X|$. The base case $|X| = 1$ is trivial – the girl can get married if she knows a boy.

Now consider a graph G with $|X| = a$ girls where Hall's condition holds, and assume the theorem holds for all smaller values of $|X|$. We split into two cases.

First, suppose Hall's condition is loose, in the sense that

$$|N(A)| \geq |A| + 1 \qquad \text{for all } A \subseteq X.$$

Then pick an edge xy, marry girl x to boy y, and delete the vertices. Since each $|N(A)|$ can only have reduced by 1, Hall's condition still holds on the remaining graph with $a - 1$ girls. Hence, by induction, the remaining girls can be married off, and we have a matching.

Second, suppose Hall's condition is tight, in that

$$|N(A)| = |A| = i \qquad \text{for some } A \subseteq X.$$

Since Hall's condition must hold for the graph restricted to the set A of i girls, those i girls can be marries off to set $N(A)$ of i boys, by induction. If we can show that Hall's condition holds for the remaining girls $X \setminus A$, then they can also be married off, by induction, and we're done.

Suppose, seeking a contradiction, that there is a set of j remaining girls that collectively know fewer than j remaining boys. Then taking these j girls together with the earlier i girls gives a set of $i + j$ girls knowing fewer than $i + j$ boys (the i boys in $N(A)$ and the fewer than j boys in the remaining graph). This contradicts Hall's condition holding in the original graph G.

This gives the result. □

Next time: *We look at graphs that can be drawn without any of the edges crossing.*

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