Topics in Discrete Mathematics
Part 2: Introduction to Graph Theory

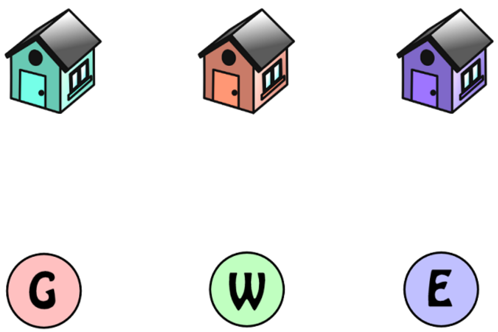
Lecture 4

Planar Graphs

- Definitions: planar graph, embeddings, faces
- Euler's formula
- The $3n-6$ rule and the nonplanarity of K_5
- Generalisation of the $3n-6$ rule and the nonplanarity of $K_{3,3}$
- Kuratowski's theorem (without proof)

4.1 The utility puzzle

Puzzle. Can you connect the three houses to the three utilities – gas, water, and electricity – without any of the pipes or wires crossing?



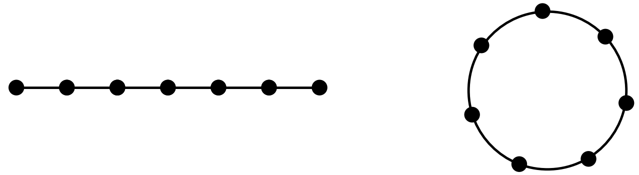
4.2 Definitions

This lecture, we're thinking about graphs that can be drawn on a piece of paper without edges crossing.

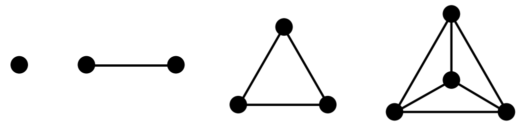
Definition 4.1. A graph G is *planar* if it can be drawn in the plane without edge crossings.

A particular drawing of a planar graph G is called a *planar embedding* of G.

So to show a graph is planar, it suffices to give a drawing (embedding) of it. So we see, for example, that the paths P_n and cycle C_n are always planar.



We also see that K_1, K_2, K_3 and (being a little bit clever with the drawing) K_4 are all planar.

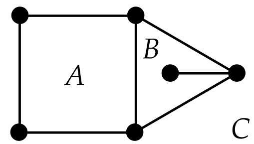


The areas in between edges are called *faces*.

Definition 4.2. An embedding of a planar graph partions the plane into a finite number of area, which we call *faces*.

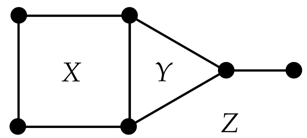
The *degree* $d(F)$ of a face F is the number of edges encountered in a walk around the perimeter of the graph.

So the following graph has three faces: A, B and the infinite face on the outside, C.



(Never forget the infinite face!) These faces have degrees $d(A) = 4$, $d(B) = 5$ and $d(C) = 5$. Note that we get $d(B) = 5$ as the edge protruding into the triangle is counted once on the way in and then once more on the way out.

Note that we could choose to draw the above graph in a different way as:



This again has three faces, X, Y, Z, but these have different degrees to before: $d(X) = 4$, $d(Y) = 3$ and $d(Z) = 7$ (again remembering we go up and down the hanging edge).

Note that whenever there is a cycle in a graph, those edges split the 'outside' of the cycle from the 'inside', so all the edges have their two sides adjacent to different faces. Conversely, edges not in any cycles, 'hang into' a face, and have that same face on both sides.

4.3 Euler's formula

Although the two drawings of the graph above had some different features, they both had the same number of faces, 3. In fact this is always the case, as Euler's formula (also known as Euler's polyhedron formula) shows.

Theorem 4.3 (Euler's formula)**.** *Consider a connected planar graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges, and an embedding of G with f faces. Then $n - m + f = 2$.*

So if a graph is planar, however we draw it, it will have $f = 2 - n + m$ faces.

We can check that Euler's formula holds. So for K_4, for example, we have $n = 4$, $m = 6$, $f = 4$, and indeed $n - m + f = 4 - 6 + 4 = 2$. (This kind of quick example is a good sanity check that you've remembered the formula correctly.)

The use of Euler's formula is that it allows you to eliminate one of the variables n, m, f in any equation – this will come in useful later.

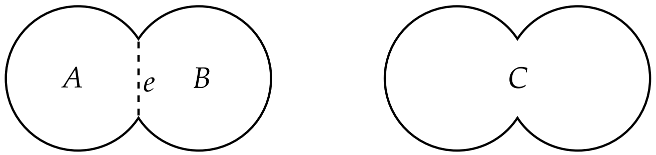
Proof. We work by induction on the number of faces f.

For the base case, if $f = 1$, then our graph has no cycles, and since it is also connected, it must be a tree. But by Theorem 2.11, we known that $m = n - 1$ for a tree. Hence

$$n - m + f = n - (n - 1) + 1 = 2,$$

as desired.

Now consider a graph with $f \geq 2$ faces, and assume the theorem holds when there are $f - 1$ faces. Since there is more than one face, our graph must contain a cycle – pick an edge e in that cycle. Removing the edge merges the two faces on either side of e, giving a graph that still has n vertices, but $m - 1$ edges and $f - 1$ faces. By the inductive hypothesis, we have $n - (m - 1) + (f - 1) = 2$, and rearranging gives $n - m + f = 2$, as desired.



□

For disconnected graphs, we have a similar result.

Theorem 4.4. *Consider a planar graph $G = (V, E)$ with c connected components $n = |V|$ vertices and $m = |E|$ edges, and an embedding of G with f faces. Then $n - m + f = 1 + c$. Generally, we have $n - m + f \geq 2$.*

Sketch proof. A simple induction on c. Adding an extra edge between two components reduces c by 1 and increases m by 1. □

Although our example above had two drawings with faces of different degrees, in fact the sum of the degrees was the same. This is not a coincidence either.

Theorem 4.5 (Handshake lemma for faces)**.** *Consider an embedding of a planar graph $G = (V, E)$ with $m = |E|$ edges. Then*

$$\sum_{\text{faces } F} d(F) = 2m.$$

The proof is very similar to the usual handshake lemma.

Proof. Go through the faces of the embedding of G, summing the edges surrounding each face – this is $\sum_F d(F)$. But each edge has been counted twice, once for either side (in different faces if the edge was in a cycle, twice in the same face if it wasn't). This is $2m$. □

4.4 Proving nonplanarity

Proving a graph is planar is easy – just draw it without edge crossings. However, showing a graph is nonplanar can be hard – just because you haven't managed to draw it, it doesn't mean that a clever embedding can't exist.

For example, you may suspect that K_5 is likely to be nonplanar. Similarly, if you've failed to solve the utility puzzle above, you've perhaps realised that showing no solutation exists would be equivalent to proving that the complete bipartite graph $K_{3,3}$ is nonplanar.

The main way we show graphs are nonplanar is by showing they have too many edges. It turns out that planar graphs have very few edges (they are 'sparse'), and so any graph with lots of edges will fail to be planar.

More specifically, we have the '$3n - 6$' rule.

Theorem 4.6. *Let $G = (V, E)$ be a planar graph with $n = |V|$ vertices and $m = |E| \geq 2$ edges. Then $m \leq 3n - 6$.*

So if a graph has more than $3n - 6$ edges it must be nonplanar. Note that $3n - 6$ edges is very few compared to, say, the complete graph K_n, which has $n(n-1)/2 \approx \frac{1}{2}n^2$ edges.

The theorem above is sufficent to prove that K_5 is indeed nonplanar, since $n = 5$, so $3n - 6 = 9$, where as K_5 has $m = 10 > 9$ edges. Hence K_5 has too many edges, breaks the $3n - 6$ rule, so must be nonplanar.

Proof. Note that every face must be surrounded by a cycle (since we've ruled out K_2 through demanding $m \geq 2$), and the shortest cycle is the triangle, of length 3. Hence the degree $d(F)$ of every face must be at least 3. Hence, by the handshake lemma for faces, we have

$$2m = \sum_{\text{faces } F} d(F) \geq \sum_{\text{faces } F} 3 = 3f.$$

Since the theorem has no f in it, we can substitute in Euler's formula to get rid of the f, getting

$$2m \geq 3(2 - n + m) = 6 - 3n + 3m.$$

Rearranging gives $m \leq 3n - 6$, as desired. □

Note that the $3n - 6$ rule is not sufficient to prove that $K_{3,3}$ is nonplanar. This has $n = 6$, so $3n - 6 = 12$, but has $m = 9$ edges, which is not so many that the rule can prove nonplanarity.

However, we can improve the theorem for graphs that avoid certain short cycles.

Theorem 4.7. *Let $G = (V, E)$ be a planar graph that has no cycles shorter than length k, with $n = |V|$ vertices and $m = |E| \geq k/2$ edges. Then*

$$m \leq \frac{k}{k-2}(n-2).$$

Note that the special case $k = 3$ gives the $3n - 6$ rule.

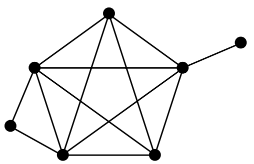
This result is strong enough to prove that $K_{3,3}$ is nonplanar. Note that $K_{3,3}$, like all bipartite graphs, is triangle-free, in that it has no 3-cycles. Hence, we can take $k = 4$ in the above theorem, to make the condition for nonplanarity that

$$m > \frac{4}{4-2}(n-2) = 2n - 4.$$

But for $K_{3,3}$, we have $n = 6$, so $2n - 4 = 8$, but $m = 9$. This is too many, and $K_{3,3}$ must be nonplanar.

Proof. The proof is the same as for the $3n - 6$ rule. Note now that every face has degree at least k, so the handshake lemma gives $2m \geq kf$. Substituting in Euler's formula gives the result. □

It's easy to see that if a graph G is nonplanar, then so is any graph that has G as a subgraph. So, for example, this graph

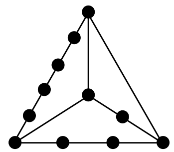


is clearly nonplanar, as the K_5 subgraph is nonplanar.

A similar idea is that of *subdividing*.

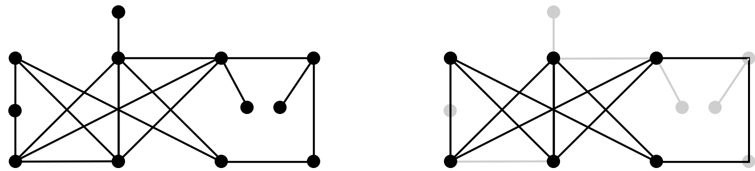
Definition 4.8. A graph H is a *subdivision* of a graph G if one can produce H from G by replacing edges of G by (disjoint) paths.

So, for example, the graph below is a subdivision of K_4.



It's easy to see that the subdivision of a nonplanar graph must be nonplanar also. Otherwise, the embedding of the path of the subdivision could be used as the embedding of the edge in the original graph.

Hence, we can show that the following graph is nonplanar by identifying a subgraph that is a subdivision of $K_{3,3}$.



We've concentrated on K_5 and $K_{3,3}$ as nonplanar graphs in this lecture. But in fact, they are, in some sense the only two nonplanar graphs there are. Kuratowski's theorem tells us that any nonplanar graph has either K_5 or $K_{3,3}$ hidden inside it.

Theorem 4.9 (Kuratowski's theorem)**.** *A graph is nonplanar if and only if it contains a subgraph that is (isomorphic to) a subdivision of either K_5 or $K_{3,3}$.*

The 'if' is clear by the discussion above; the 'only if' is difficult and outside the bounds of this course.

Next time: *Using linear algebra in graph theory (so you might want to remind yourselves about eigenvalues and things like that).*

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