Topics in Discrete Mathematics

Part 2: Introduction to Graph Theory

Problem Sheet

*Questions **1**, **4**, **6**, **8**, and **11** due in by 1000, Thursday 20 February.*
There's a small prize for the best score on 10(b).
Other questions are optional (but recommended).

1. (a) For each of the following sequences (x_n), either give a graph with degree sequence (x_n), or explain why no such graph exists:
   i. $(4, 2, 2, 1, 1)$,
   ii. $(2, 2, 2, 0)$,
   iii. $(3, 2, 2, 2)$,
   iv. $(4, 2, 1, 1)$
   v. $(3, 3, 3, 1)$,
   vi. $(1, 1, \dots)$ with n repeated 1s,
   vii. $(n-1, n-2, \dots, 2, 1, 0)$.

   (b) Prove that every nontrivial graph has two vertices of the same degree.

   (c) Prove or give a counterexample: any two graphs with the same degree sequence are isomorphic.

2. The *complement* $\overline{G}$ of a graph $G = (V, E)$ is defined by $\overline{G} = (V, V^{(2)} \setminus E)$. In other words, we keep the the same vertices, but replace nonedges with edges, and edges with nonedges.

   (a) What is the complement of
   i. the complete graph K_n,
   ii. the complete bipartite graph $K_{a,b}$,
   iii. the path P_4,
   iv. the cycle C_5?

   (b) i. Why might one describe P_4 and C_5 as 'self-complementary'?
   ii. If a graph is self-complementary and has n vertices, how many edges does it have?
   iii. Show that a graph with n vertices can only be self-complementary if $n = 0$ or $1 \pmod 4$.

   (c) If A and $\overline{\mathsf{A}}$ are the adjacency matrices of G and $\overline{G}$ respectively, what is $\mathsf{A} + \overline{\mathsf{A}}$?

   (d) Show that either a graph or its complement are connected.

3. Fix a graph $G = (V, E)$. For $u, v \in V$, write $d(u, v)$ for the distance of the shortest path from u to v. If $u = v$, write $d(u, v) = 0$; and if u and v are in different connected components of the graph, write $d(u, v) = \infty$.

   (a) Prove that d is a metric on V.

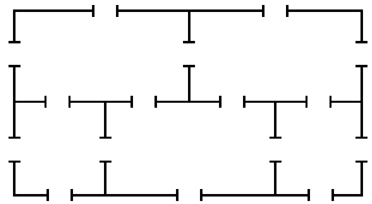
   (b) Define, as formally as you can, the *Erdős number*. (*Hint:* `en.wikipedia.org/wiki/Erdos_number`)

   (c) i. Give a graph where $V = \{0, 1\}^k$ binary vectors of length k and the metric d is the Hamming distance.
   ii. Show that your graph is bipartite.
   iii. Give a matching.

4. (a) i. For what values of n does K_n have an Eulerian circuit?
   ii. For what values of a and b does $K_{a,b}$ have an Eulerian circuit?

   (b) Repeat part (a) for Hamiltonian cycles.

   (c) Can you walk through every doorway in this building, ending up in the room you started in?

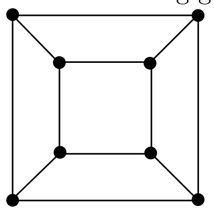


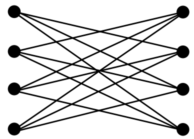
5. (a) Give a necessary and sufficient condition for a connected graph to have an Eulerian circuit.

   (b) An *Eulerian trail* is a trail containing every edge of a graph. Prove that a connected graph has an Eulerian trail (that isn't an Eulerian circuit) if and only if it has two vertices odd degree and the remaining vertices all have even degree.

   (c) What can you say of a connected graph with exactly one vertex of odd degree?

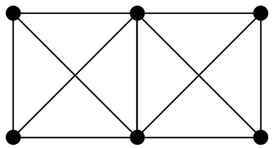
6. (a) Show that the following graphs are isomorphic.





   (b) Do they have an Eulerian circuit?
   (c) Do they have a Hamiltonian cycle?
   (d) Show they have a matching by
      i. checking that Hall's condition holds,
      ii. exhibiting a matching.

7. Prove that a graph is bipartite if and only if it has no odd cycles.

8. (a) Which of the following graphs are planar? Prove each graph is nonplanar, or draw it without edge crossings.
      i. 
      ii. the complete graph K_n,
      iii. the complete bipartite graph $K_{a,b}$
   (b) Give two different proofs that the Petersen graph is nonplanar.

9. By considering the minimum number of edges surrounding each face and the number of faces adjacent to each edge, show that, for a planar graph G with $m \geq 2$ edges, we have $2m \geq 3f$, where f is the number of faces.

10. Play the game Planarity at `planarity.net`.
   (a) Explain clearly, using terminology from the course, what you are doing.
   (b) What's the highest score you managed to get? (A small prize for the highest score in the class.)

11. (a) Calculate the spectra of
      i. $K_2 = K_{1,1}$,
      ii. $P_3 = K_{1,2}$,
      iii. $C_4 = K_{2,2}$.
   (b) Show that the spectrum of any bipartite graph is symmetric (in that, if λ is an eigenvalue, then so is $-\lambda$).

12. To *orient* a graph means to label each of its edges with an arrow. Show that any graph G can be oriented such that, for each vertex v, the number of edges pointing into v and the number of edges pointing out of v differ by at most 1. (*Hint:* Can you prove it if G is a cycle? Can you prove it if G is a tree.)

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