Topics in Discrete Mathematics
Part 2: Introduction to Graph Theory

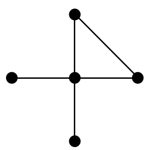
Solutions to Problem Sheet

I've provided full solutions for the set questions,
and partial solutions or hints for the remaining questions.

1. (a) For each of the following sequences (x_n), either give a graph with degree sequence (x_n), or explain why no such graph exists:

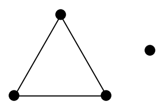
i. $(4, 2, 2, 1, 1)$,

Solution:



ii. $(2, 2, 2, 0)$,

Solution: $K_3 \cup K_1$



iii. $(3, 2, 2, 2)$,

Solution: Not possible: by the handshaking lemma, the sum of the degrees must be even.

iv. $(4, 2, 1, 1)$,

Solution: Not possible: the maximum allowable degree in a 4-vertex graph is 3.

v. $(3, 3, 3, 1)$,

Solution: Not possible: the three 3s must be adjacent to every other vertex, but the 1 won't allow this.

vi. $(1, 1, \dots)$ with n repeated 1s,

Solution: When n is odd: not possible by the handshaking argument.
When n is even: a union of $n/2$ copies of K_2.

vii. $(n - 1, n - 2, \dots, 2, 1, 0)$.

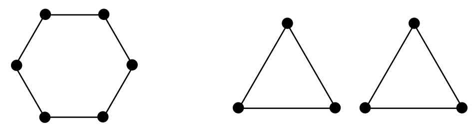
Solution: Not possible: the $n - 1$ wants to be adjacent to every other vertex, but the 0 wants to be joined to no other vertex – these can't both happen.

(b) Prove that every nontrivial graph has two vertices of the same degree.

Solution: Write $n \geq 2$ for the number of vertices. Since the minimum allowed degree is 0 and the maximum allowed is $n-1$, the only way we could avoid repeats is by having degree sequence $(n-1, n-2, \dots, 2, 1, 0)$. But by (a) vii. above, this isn't possible.

(c) Prove or give a counterexample: any two graphs with the same degree sequence are isomorphic.

Solution: One counterexample is $(2, 2, 2, 2, 2, 2)$, which can be C_6 or $C_3 \cup C_3$.



2. The *complement* $\overline{G}$ of a graph $G = (V, E)$ is defined by $\overline{G} = (V, V^{(2)} \setminus E)$. In other words, we keep the the same vertices, but replace nonedges with edges, and edges with nonedges.

(a) What is the complement of

i. the complete graph K_n,

Solution: The empty graph with no edges on n vertices.

ii. the complete bipartite graph $K_{a,b}$,

iii. the path P_4,

iv. the cycle C_5?

(b) i. Why might one describe P_4 and C_5 as 'self-complementary'?

> **Solution:** Their complement is isomorphic to themself.

ii. If a graph is self-complementary and has n vertices, how many edges does it have?

> **Hint:** Half the number of the complete graph. (Why?)

iii. Show that a graph with n vertices can only be self-complementary if $n = 0$ or $1 \pmod 4$.

> **Hint:** Your previous answer must be an integer.

(c) If A and $\overline{\mathsf{A}}$ are the adjacency matrices of G and $\overline{G}$ respectively, what is $\mathsf{A} + \overline{\mathsf{A}}$?

(d) Show that either a graph or its complement are connected.

> **Hint:** Try starting like this:
> *Let $G = (V, E)$ be a disconnected graph, and let $\overline{G}$ be its complement. We want to show that $\overline{G}$ is connected, so take vertices $u, v \in V$, between which we want to find a walk in $\overline{G}$.*
> *If $uv \notin V$ is not an edge in G, then....*
> *If $uv \in V$ is an edge in G, then....* (Use the fact that G is disconnected.)

3. Fix a graph $G = (V, E)$. For $u, v \in V$, write $d(u, v)$ for the distance of the shortest path from u to v. If $u = v$, write $d(u, v) = 0$; and if u and v are in different connected components of the graph, write $d(u, v) = \infty$.

(a) Prove that d is a metric on V.

> **Hint:** Only the triangle inequality should require any thought. How would breaking the triangle inequality contradict 'shortest' path?

(b) Define, as formally as you can, the *Erdős number*. (*Hint:* `en.wikipedia.org/wiki/Erdos_number`)

> **Hint:** Don't worry about this – it's a silly question.

(c) i. Give a graph where $V = \{0, 1\}^k$ binary vectors of length k and the metric d is the Hamming distance.

> **Hint:** For $k = 3$, take the cube.
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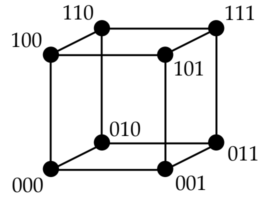
> 
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>
> Can you generalise?

ii. Show that your graph is bipartite.

> **Hint:** Consider the weights $w(\mathbf{x}) = d(\mathbf{x}, \mathbf{0})$. What can you say about the weights of adjacent vertices?

iii. Give a matching.

> **Hint:** What happens if you truncate the final coordinate of your vectors?

4. (a) i. For what values of n does K_n have an Eulerian circuit?

> **Solution:** Since K_n is connected, we just need all vertices to have even degree. Since all the degrees are $n - 1$, an Eulerian circuit exists when (and only when) n is odd.

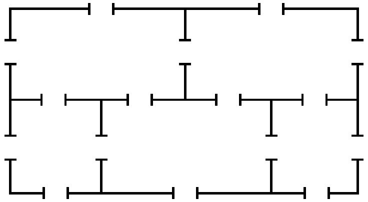
ii. For what values of a and b does $K_{a,b}$ have an Eulerian circuit?

> **Solution:** By the same logic, we have an Eulerian circuit whe all degrees are even, which is when both a and b are even.

(b) Repeat part (a) for Hamiltonian cycles.

> **Solution:**
> i. Neither K_1 nor K_2 have any cycle. But for $n \geq 3$, it's easy to see that $123\ldots(n-1)n1$ is a Hamiltonian cycle.
>
> ii. Any cycle will go left-right-left-right-etc. across the bipartition. Hence we can only possible have a Hamiltonian cycle when $a = b$. Clearly $a = b = 1$ won't work. But for $a = b \geq 2$, we fulfil the conditions of Dirac's theorem, so have a cycle.

(c) **Correction:** Can you walk through every doorway in this building ***exactly once***, ending up in the room you started in?



> **Solution:** As with the bridges of Königsberg problem, we seek an Eulerian circuit in the following graph.
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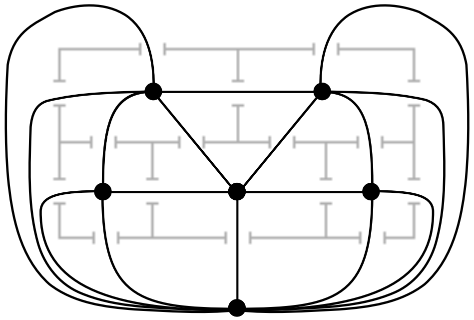
> 
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> (If you're worried about multiple edges, subdivide some of them.) Clearly we have many odd-degree vertices, so no Eulerian circuit exists.

5. (a) Give a necessary and sufficient condition for a connected graph to have an Eulerian circuit.

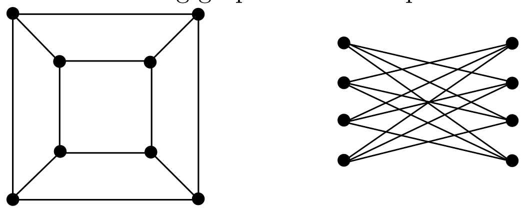
   (b) An *Eulerian trail* is a trail containing every edge of a graph. Prove that a connected graph has an Eulerian trail (that isn't an Eulerian circuit) if and only if it has two vertices odd degree and the remaining vertices all have even degree.

> **Hint:** Call the odd-degree vertices u and v. Clearly an Eulerian trail must start at u and end at v (or vice versa). From here you two choices.
>
> One is to go through the proof of part (a) from lectures, adapting it appropriately.
>
> The other is to add a new vertex w and edges vw and wu. What can you say about this new graph?

   (c) What can you say of a connected graph with exactly one vertex of odd degree?

> **Hint:** This is a trick question.

6. (a) Show that the following graphs are isomorphic.



> **Solution:** This is easy to see if we label the graphs as follows:
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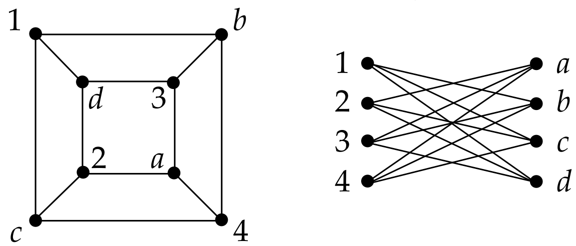
> 
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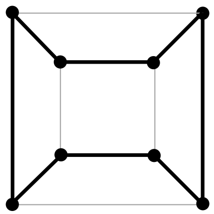
(b) Do they have an Eulerian circuit?

Solution: No: there are many vertices of odd degree.

(c) Do they have a Hamiltonian cycle?

Solution: Yes: see picture for one example.



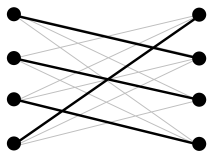
(d) Show they have a matching by

i. checking that Hall's condition holds,

Solution: Let's work on the second picture. Any single vertex on the left is connected to three vertices on the right; two or more vertices on the left are connected to all four on the right. Hence Hall's condition holds, and a matching exists.

ii. exhibiting a matching.

Solution:



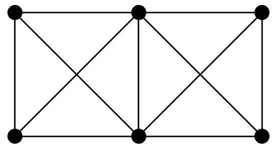
7. Prove that a graph is bipartite if and only if it has no odd cycles.

Hint: Try starting like this:

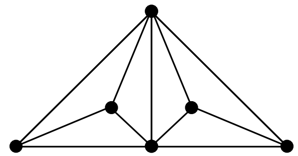
(Only if) Suppose G is bipartite, and consider a cycle $v_0 v_1 v_2 \cdots v_k v_0$. We want to show that k is even. ...

(If) Suppose G has no cycles, and assume, without loss of generality, that G is connected. Then we can create a bipartition as follows. Pick a starting vertex v. Then....

8. (a) Which of the following graphs are planar? Prove each graph is nonplanar, or draw it without edge crossings.

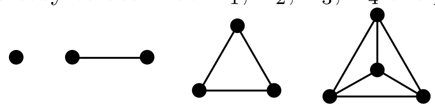
i. 

Solution: Planar.



ii. the complete graph K_n,

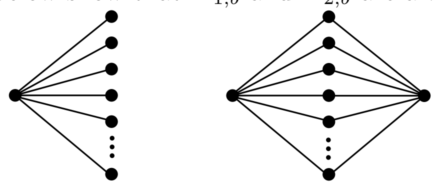
Solution: It's easy to see that K_1, K_2, K_3, K_4 are planar.



We know K_5 is nonplanar, as it fails the $3n-6$ rule. For $n \geq 5$, K_n is nonplanar, as it has K_5 as a nonplanar subgraph.

iii. the complete bipartite graph $K_{a,b}$.

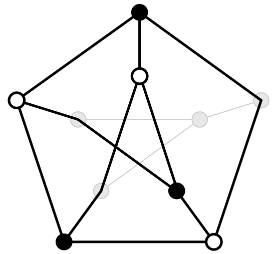
Solution: Since $K_{a,b} = K_{b,a}$, we may take $a \leq b$ without loss of generality.
The pictures below show that $K_{1,b}$ and $K_{2,b}$ are always planar.



We know $K_{3,3}$ is nonplanar, as it fails the $2n-4$ rule for triangle-free graphs. For $b \geq a \geq 3$, $K_{a,b}$ is nonplanar, as it has $K_{3,3}$ as a nonplanar subgraph.

(b) Give two different proofs that the Petersen graph is nonplanar.

Solution: First, the picture below shows a subgraph that is a subdivion of $K_{3,3}$. This shows that the Petersen graph is nonplanar, by Kuratowski's theorem.



Second, recall the theorem that, for a planar graph with no cycles shorter than length k, we have

$$m \leq \frac{k}{k-2}(n-2),$$

provided that $n \geq k/2$. Note that the Petersen graph has no 3-cycles (triangles) or 4-cycles (squares). Hence we can take $k = 5$ in the theorem and see that the Petersen graph could only be planar if

$$m \leq \frac{5}{5-2}(n-2) = \tfrac{5}{3}n - \tfrac{10}{3}.$$

For the Petersen graph, $n = 10$, so $\frac{5}{3}n - \frac{10}{3} = 13\frac{1}{3}$. But in fact $m = 15$, so the Petersen graph has too many edges and cannot be planar.

9. By considering the minimum number of edges surrounding each face and the number of faces adjacent to each edge, show that, for a planar graph G with $m \geq 2$ edges, we have $2m \geq 3f$, where f is the number of faces.

Hint: See the proof of the $3n - 6$ rule in the lecture notes.

10. Play the game Planarity at `planarity.net`.
    (a) Explain clearly, using terminology from the course, what you are doing.
    (b) What's the highest score you managed to get? (A small prize for the highest score in the class.)

11. (a) Calculate the spectra of
    i. $K_2 = K_{1,1}$,

Solution: The adjacency matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solving the characteristic equation gives

$$0 = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1).$$

So the spectrum is $\lambda = 1, -1$.

ii. $P_3 = K_{1,2}$,

Solution: The adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solving the characteristic equation gives

$$0 = \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix} = \lambda(\lambda^2 - 1) + \lambda = \lambda(\lambda - \sqrt{2})(\lambda + \sqrt{2}).$$

So the spectrum is $\lambda = \sqrt{2}, 0, -\sqrt{2}$.

iii. $C_4 = K_{2,2}$.

Solution: The adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The matrix has rank 2, since the bottom half of the maatrix repeats the top half,so 0 must be an eigenvalue twice. It is also easy to see that $(1, 1, 1, 1)$ is an eigenvector with eigenvalue 2, and that $(1, -1, 1, -1)$ is an eigenvector with eigenvalue -2. So the spectrum is $\lambda = 2, 0, 0, -2$.

(b) Show that the spectrum of any bipartite graph is symmetric (in that, if λ is an eigenvalue, then so is $-\lambda$).

Solution: Order the vertices so that all those on the left-hand side of the bipartition come before those on the right-hand side. Then the adjacency matrix can be written in block form as

$$\mathsf{A} = \begin{pmatrix} \mathsf{0} & \mathsf{B} \\ \mathsf{B}^\top & \mathsf{0} \end{pmatrix}.$$

Suppose $\mathbf{x}$ is an eigenvalue of A with eigenvalue λ, and write $\mathbf{x}$ in the same block form as $\mathbf{x} = (\mathbf{y} \quad \mathbf{z})^\top$. Then we have

$$\mathsf{A} \begin{pmatrix} -\mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathsf{0} & \mathsf{B} \\ \mathsf{B}^\top & \mathsf{0} \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathsf{B}^\top \mathbf{z} \\ \mathsf{B}\mathbf{y} \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{y} \\ \lambda\mathbf{z} \end{pmatrix}.$$

But now consider the vector $(-\mathbf{y} \quad \mathbf{z})^\top$. We have that

$$\mathsf{A} \begin{pmatrix} -\mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathsf{0} & \mathsf{B} \\ \mathsf{B}^\top & \mathsf{0} \end{pmatrix} \begin{pmatrix} -\mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathsf{B}^\top \mathbf{z} \\ -\mathsf{B}\mathbf{y} \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{y} \\ -\lambda\mathbf{z} \end{pmatrix} = -\lambda \begin{pmatrix} -\mathbf{y} \\ \mathbf{z} \end{pmatrix}.$$

So $(-\mathbf{y} \quad \mathbf{z})^\top$ is an eigenvector with eigenvalue $-\lambda$.
This proves the result.

12. To *orient* a graph means to label each of its edges with an arrow. Show that any graph G can be oriented such that, for each vertex v, the number of edges pointing into v and the number of edges pointing out of v differ by at most 1. (*Hint:* Can you prove it if G is a cycle? Can you prove it if G is a tree.)

Hint: This question is pretty hard, so don't worry about it too much. But the solution is in *The Art of Mathematics* by Béla Bollobás, if you want it.

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