Lecture 6
Constructing measures III: Carathéodory’s extension theorem

- Premeasures and semialgebras
- Carathéodory’s extension theorem for algebras and for semialgebras.
- Lebesgue measure: existence and uniqueness

6.1 Carathéodory’s extension theorem for algebras

The story so far:

0. We started with a set \( X \) and a collection \( \mathcal{R} \) of sets, with a function \( \rho \) on \( \mathcal{R} \), representing the sets whose measure we ‘know’.

1. We constructed an outer measure \( \mu^* \) on all of \( \mathcal{P}(X) \).

2. We saw that if we restrict \( \mu^* \) to \( \mu \) on just the measurable sets \( \mathcal{M} \) (satisfying the splitting condition), then \( \mu \) is a measure on \( \mathcal{M} \).

However, we haven’t guaranteed that this constructed measure \( \mu \) extends \( \rho \), in the sense that all of \( \mathcal{R} \) is measurable, and \( \mu(R) = \rho(R) \) for \( R \in \mathcal{R} \). In fact, in general, it’s not true. However, it is true if \( \mathcal{R} \) and \( \rho \) have certain properties.

First let’s deal with \( \rho \). Clearly \( \rho \) can’t by itself contradict the measure axioms, as then an extension would have no hope.

**Definition 6.1.** Let \( X \) be a nonempty set, and let \( \mathcal{R} \) be collection of subsets of \( X \) containing \( \emptyset \). Then a function \( \pi: \mathcal{R} \to [0, \infty] \) is a **premeasure** on \( (X, \mathcal{R}) \) if

1. \( \pi(\emptyset) = 0; \)

2. if \( A_1, A_2, \ldots \) is a countable sequence of disjoint sets in \( \mathcal{R} \) and if their union \( \bigcup_{n=1}^{\infty} A_n \) is also in \( \mathcal{R} \), then

\[
\pi\left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \pi(A_n).
\]

We will also need \( \mathcal{R} \) to have some of the structure of a \( \sigma \)-algebra. An algebra is an example of this we already know.

**Theorem 6.2** (Carathéodory’s extension theorem for algebras). Let \( X \) be a nonempty set, \( \mathcal{A} \) be an algebra on \( X \), and \( \pi \) a premeasure on \( (X, \mathcal{A}) \). Then there exists a measure \( \mu \) which extends \( \pi \), in the sense that \( \mu \) is a measure on \( (X, \sigma(\mathcal{A})) \) with \( \mu(A) = \pi(A) \) for \( A \in \mathcal{A} \).

Further, if \( \mu \) is a \( \sigma \)-finite measure, then it is the unique such extension.

The idea is to take \( \mu \) to be the restriction of the outer measure \( \mu^* \) constructed via the covering method. So to prove the extension theorem we need to show that

- \( \sigma(\mathcal{A}) \subset \mathcal{M} \), the measurable sets – since \( \mathcal{M} \) is a \( \sigma \)-algebra, just showing \( \mathcal{A} \subset \mathcal{M} \) suffices;

- \( \mu(A) = \pi(A) \) for \( A \in \mathcal{A} \);

- uniqueness (in the \( \sigma \)-finite case).

We shall do the proof later.

Since \( \mathcal{M} \) is a complete measure space (see Problem Sheet 3), we could extend \( \pi \) even further to the completion \( (X, \sigma(\mathcal{A}), \bar{\mu}) \) of \( (X, \sigma(\mathcal{A}), \mu) \) if we wished, but we won’t bother in this course.

6.2 Carathéodory’s extension theorem for semialgebras

While Theorem 6.2 is an important result, asking for \( \mathcal{R} \) to be an algebra is quite a strenuous requirement. For example, the collection of intervals \( \mathcal{I} \) we have in the setup for the Lebesgue measure is not an algebra, so this theorem is insufficient to prove existence of the Lebesgue measure.

Instead, we will look at a weaker definition.

**Definition 6.3.** Let \( X \) be a nonempty set, and \( \mathcal{S} \) a collection of subsets of \( X \). Then \( \mathcal{S} \) is a **semialgebra** if

1. \( \emptyset \in \mathcal{S} \);

2. \( \mathcal{S} \) is closed under finite intersections, in that for \( A, B \in \mathcal{S} \) we have \( A \cap B \in \mathcal{S} \);

3. ‘complements are finite disjoint unions,’ in that for \( A \in \mathcal{S} \), there exists disjoint \( B_1, B_2, \ldots, B_N \) in \( \mathcal{S} \) such that \( A^c = \bigcup_{n=1}^{N} B_n \).
Theorem 6.4. Every algebra is a semialgebra.

Proof. The empty set is immediate, complements as finite disjoint unions from setting $B_1 = A$, and intersections follows from De Morgan’s law.

Theorem 6.5 (Carathéodory’s extension theorem for semialgebras). Let $X$ be a nonempty set, $S$ be a semialgebra on $X$, and $\pi$ a premeasure on $(X,S)$. Then there exists a measure $\mu$ which extends $\pi$.

Further, if $\mu$ is a $\sigma$-finite measure, then it is the unique such extension.

Again, we postpone the proof.

6.3 The Lebesgue measure

Back in Lecture 3, we defined the Lebesgue measure on $\mathbb{R}$ as follows.

Definition 6.6. The Lebesgue measure on $\mathbb{R}$ is the unique measure $\lambda$ on $(\mathbb{R},\mathcal{B})$ such that $\lambda([a,b]) = b-a$ for all $a \leq b$.

The Lebesgue measure on $\mathbb{R}^d$ is the unique measure $\lambda$ on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ such that

$$\lambda\left(\prod_{i=1}^{d}([a_i,b_i])\right) = \prod_{i=1}^{d}(b_i-a_i) \quad \text{for all } a_i < b_i, \ i = 1,2,\ldots,d.$$

Theorem 6.7. The Lebesgue measure exists and is unique.

We shall give the full proof just for the $d=1$ case, although the general case is much the same. Also the product measure (see Problem Sheet 3, and later in the course) gives an alternative construction in the $d \geq 2$ case.

Proof. First some housekeeping. All the intervals $[a,b]$ must be measurable, as must the empty set, while taking countable unions shows the infinite intervals $(\infty,b),[a,\infty),\mathbb{R}$ must be measurable too. This gives all the intervals in $\mathcal{I}$. Further, by countable additivity, the infinite intervals must have measure $\infty$, and the empty set must have measure 0. This gives the length function $\rho$ as defined in the ‘setup’ of Lecture 4.

Since $\sigma(\mathcal{I}) = \mathcal{B}$, Carathéodory’s extension theorem will show that the restriction the Lebesgue outer measure $\lambda^*$ to $\mathcal{B}$ works. We just need to show that $\mathcal{I}$ is a $\sigma$-algebra, that the length $\rho$ is a premeasure, and that $\lambda$ is $\sigma$-finite. We prove these in the upcoming lemmas.

Lemma 6.8. The collection of intervals $\mathcal{I}$ is a semialgebra on $\mathbb{R}$.

Proof. That $\emptyset \in \mathcal{I}$ is immediate. For finite intersections, note that

$$[a,b) \cap [c,d) = \left[\max(a,c),\min(b,d)\right).$$

(with the latter interval interpreted as $\emptyset$ where necessary), with a similar result for the infinite and empty intervals. For complements, we have $[a,b]^c = (\infty,a] \cup [b,\infty)$, and similar for the infinite and empty intervals.

Basically the same proof works in $d$ dimensions, although it takes a few lines to show the complement of an interval box in $\mathbb{R}^d$ can be written as a union of (at most) $2d$ interval boxes.

Lemma 6.9. The ‘length’ function $\rho$ is a premeasure on $(\mathbb{R},\mathcal{I})$.

Proof. We certainly have $\rho(\emptyset) = 0$.

We need to show countable additivity. Let $I_1,I_2,\ldots$ be disjoint intervals whose union is also an interval $I$. For finite $N$, we clearly have $\rho(I) \geq \sum_{n=1}^{N} \rho(I_n)$, for example by sorting the intervals from left to right. Sending $N \to \infty$ gives $\rho(I) \geq \sum_{n=1}^{\infty} \rho(I_n)$.

We have to show the inequality the other way.

First, we assume the intervals are non-infinite, so the $I_n = [a_n,b_n)$ are disjoint, and their union is $I = \cup_i [a_i,b_i)$. We use a compactness argument, which allows us to reduce from infinitely many to finitely many sets. Extend each interval to an open interval $I_n' = (a_n-\epsilon/2^n,b_n)$ (noting an upcoming use of the $\epsilon/2^n$ trick). Then these intervals cover the compact interval $[a,b-\epsilon]$ of length $b-a-\epsilon$. Since this is compact, the open cover $\{I_1',I_2',\ldots\}$ has a finite subcover $\{I_{n_1}',I_{n_2}',\ldots\}$. Thus

$$b-a-\epsilon \leq \sum_{j=1}^{k} \rho(I_{n_j}') \leq \sum_{j=1}^{k} \left(\rho(I_{n_j}) + \frac{\epsilon}{2^n}\right) \leq \sum_{j=1}^{k} \rho(I_{n_j}) + \epsilon \leq \sum_{n=1}^{\infty} \rho(I_n) + \epsilon.$$

Hence $\rho(I) \leq \sum_{n=1}^{\infty} \rho(I_n)$ and, since $\epsilon$ was arbitrary, this proves the inequality.

Suppose instead that $I$ is infinite. Then for fixed $M$, the intervals $I_n \cap [-M,M]$ are disjoint with union $I \cap [-M,M)$. By the previous paragraph, we see that

$$\sum_{n=1}^{\infty} \rho(I_n) \geq \sum_{n=1}^{\infty} \rho(I_n \cap [-M,M]) \geq \rho(I \cap [-M,M)).$$

The right-hand side tends to $\infty$ as $M \to \infty$, so the left-hand side must equal $\infty$ also. This gives the result.

Lemma 6.10. The Lebesgue measure $\lambda$ on $\mathbb{R}$ is $\sigma$-finite.

Proof. We have the countable union $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n,n)$ with $\lambda([-n,n) = 2n \leq \infty$. 

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